POWERFUL RAY PATTERNS

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I certify that I have read this thesis and certify that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Chair

Abstract

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Since the concept of ray pattern was introduced, many authors have studied properties of ray pattern. At the first appearance of ray pattern, authors considered numerical properties of complex matrices by using the concept of ray pattern. In this sense, a ray pattern can be considered as a abstraction of complex matrices. On the other hand, there had been numerous studies on combinatorial properties of sign patterns. Hence extension from sign patterns to ray patterns was very natural to get more generalized results in combinatorial matrix theory. So a ray pattern has two aspects; an abstraction of a complex matrix and a generalization of a sign pattern.

In this thesis, we are going to think about a certain combinatorial property of ray patterns. Ray patterns which we are most interested in in this thesis behave well under powers, called powerful ray patterns, in the sense that any power of a given ray pattern does not have ambiguous entries. Also we are going to consider the set S. A ray pattern is in S if it is ray diagonally similar to a ray multiple of Boolean pattern of itself. We are going to address three questions and answer them partially or fully in this thesis. Those questions are characterizing powerful ray patterns, checking powerfulness of irreducible ray patterns by powering, and characterizing the set S. The first question is still open

in general case. We are going to answer this question for ray patterns whose diagonal blocks of Frobenius normal form are primitive. For the second question, we are going to see an answer which gives us an upper bound on the first power that a non-powerful ray pattern will encounter an ambiguous entry. This answer does not cover every possible cases but exceptional cases are very specialized. For the last question, we are going to see two complete answers by using products of chains and powers of a certain matrix. Furthermore, we are going to have an algorithm that checks if a given ray pattern is in S or not by combining those two answers.

At the end of this thesis, we are going to see examples of ray patterns which are not considered in this thesis. Those examples illustrate three possible cases of ray patterns that are reducible and non-powerful. We hope that studying those three cases would lead us to a complete answer for characterizing powerful ray patterns.

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The most important reason I can be here is my father and mother. Without them, I couldn't have existed. Thanks to my parents, I have explored and enjoyed two wonderful worlds, a real world and a world of math. On my way to finish Ph.D program, I could feel their love and support from my heart even though we were half-a-world apart. Also my thanks goes to my two brothers and their wives. They wished my success as much as my parents did and gave me unconditional supports. When having hard times, words of comfort of my friends, Jungmin, Hongwon, Hwakyung, Ilyoung and Kwonbum, really helped me in mental stability. My friends are one of the most precious values to me.

I remember that the first time I stepped into Judi's office and talked to her face to face. At our first meeting, what I figured out was my good luck to have a nice advisor. She is like a kind advisor, a close friend or a generous mother who is always on my side. She helped me mentally, academically and even my campus life. I have a mother in Korea but I would like say she is my academical mother. What I learned from Judi is not only math knowledge but also relationship between human beings. Without her, I couldn't have done anything for the last three years. As a foreign student, living in U.S is not an easy going story, language, foods and life style, for examples. Friends I met here, Amy, DeAnne, Jay, Sherod and T.J, took me as I was and were like old friends I knew for ten years. I really appreciate their help, and value the memory of having fun together. Also I would like to thank all faculty members who taught me directly or indirectly and staffs of math department. Using math terminology, they were a basis for my study.

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Chapter 1

Introduction

Combinatorial matrix theory involves determining properties of matrices by looking at their underlying combinatorial structure. In particular, qualitative matrix theory seeks to determine interesting properties of a matrix that are independent of the magnitudes of the entries of the matrix. Until recently, most of this work focused on matrices over the boolean numbers, the integers, or the real numbers. In [10], McDonald, Olesky, Tsatsomeros and van den Driessche move the exploration into the complex numbers by looking at ray patterns of matrices. There are now several interesting papers on this topic (see for example [4, 5, 6, 12]).

We define a *ray pattern* to be a matrix each of whose entries is either 0 or a ray in the complex plane of the form $re^{\mathrm{i}\theta}$, where θ is a real number and r runs through all positive real numbers. For brevity, we denote a ray $re^{\mathrm{i}\theta}$ by $e^{\mathrm{i}\theta}$. For two rays $e^{\mathrm{i}\theta_1}$ and $e^{\mathrm{i}\theta_2}$, if $\theta_1 - \theta_2$ is an integer multiple of 2π, then $e^{\mathrm{i}\theta_1} = e^{\mathrm{i}\theta_2}$; otherwise, $e^{\mathrm{i}\theta_1} \neq e^{\mathrm{i}\theta_2}$. A *sign pattern* is a matrix each of whose entries is $0, -1$ or 1 and can be considered as the abstraction of real matrices. A *Boolean matrix* is a matrix whose entries are either 0 or 1 and arithmetic operations follow the rules of Boolean algebra. By simplifying $e^{\mathrm{i}0} = 1$ and $e^{\mathrm{i}\pi} = -1$, we can consider the set of Boolean matrices and the set of sign patterns as subclasses of the set of ray patterns.

Many modeling techniques examine the long run behavior of a system and this information is often contained in the powers of a matrix. Authors including Eschenbach, Hall, Li, and Stuart study the properties of powers of sign patterns (See [7, 13, 16]). It is natural to generalize sign patterns to complex ray patterns and these authors studied this topic in the recent papers [8, 14, 9].

In this thesis we look at ray patterns for which all the powers of these ray patterns are also ray patterns. Such patterns are called *powerful*. Of particular interest is the subset of the ray patterns

$$S = \{A|A \text{ is diagonally similar to } \omega|A| \text{ for some ray } \omega\}$$

The main definitions and notational conventions are contained in Chapter 2.

In Chapter 3 we look at properties of irreducible ray patterns and their powers.

We begin with a review of material from my Masters of Science work with Cho and Kim. In Section 3.1 we characterize irreducible powerful ray patterns by showing that they must be in S, and we look at periodic ray patterns more closely. For an irreducible powerful ray pattern A, let

$$\Omega(A) = \{\omega|\ A \text{ is ray diagonally similar to } \omega|A|\}$$

We show that if $\omega \in \Omega(A)$, then $e^{\frac{2m\pi i}{k}}\omega \in \Omega(A)$ where $0 \leq m \leq k$ and k is the index of imprimitivity of A. From this we see that the cardinality of $\Omega(A)$ is, in fact, k. Much of the work included in Section 3.1 has been published in [3].

We continue with new work on irreducible ray patterns in Section 3.2 by looking for an upper bound on the first power that a non-powerful matrix will encounter an ambiguous entry. In Section 3.2.2 we show that if an irreducible $n \times n$ matrix A is not powerful, then A^t contains an ambiguous entry for some $t \leq n^2 - 2n + 2$, in all but one very specialized case, which remains open. In Section 3.2.3, we show that there is a ray pattern (and sign pattern) associated with the Wielandt graph for which $t = n^2 - 2n + 2$ and hence our bound is the minimum possible.

In Chapter 4, we look at properties of powerful reducible ray patterns.

In Section 4.1, we look at the case where the diagonal blocks of the reducible ray pattern are primitive. In Section 4.2, we show that as long as none the final classes of the reducible ray pattern are trivial, then A is powerful if and only if A^k is powerful for any $k \geq 1$.

In Section 4.3, we look at two characterizations of the reducible powerful ray patterns in S. For the first characterization, we define a product of a semiwalk which is an generalized concept of a product of a walk. And then we can get a system of equations which semicycles should satisfy. Second characterization makes use of a matrix defined by $A_{(\alpha)} = A + \alpha^2 A^*$ for a ray α. For a ray pattern A of order n and a ray ω, if $(A_{(\alpha)})^{2n}$ or $(A_{(\alpha)})^{4n-6}$ is well-defined then $A \sim \alpha|A|$ or $A \sim -\alpha|A|$. The choice of powers from $2n$ and $4n-6$ depends on the existence of odd semicycle in the diagraph of a ray pattern. By combining two characterizations, we can get an algorithm which enables us to determine a given ray pattern is in S or not easily.

We conclude this thesis with a discussion of future work in Chapter 5 by considering three examples of reducible and non-powerful ray patterns. Those examples come from three classes of reducible and non-powerful ray patterns. If we can get characterizations of those classes in future, we can answer the question of characterizing powerful ray patterns in general.

Chapter 2

Notation and Definitions

We define a *ray pattern* to be a matrix each of whose entries is either 0 or a ray in the complex plane of the form $re^{\mathrm{i}\theta}$, where θ is a real number and r runs through all positive real numbers. For brevity, we denote a ray $re^{\mathrm{i}\theta}$ by $e^{\mathrm{i}\theta}$. For two rays $e^{\mathrm{i}\theta_1}$ and $e^{\mathrm{i}\theta_2}$, if $\theta_1 - \theta_2$ is an integer multiple of 2π, then $e^{\mathrm{i}\theta_1} = e^{\mathrm{i}\theta_2}$; otherwise, $e^{\mathrm{i}\theta_1} \neq e^{\mathrm{i}\theta_2}$. By simplifying $e^{\mathrm{i}0} = 1$ and $e^{\mathrm{i}\pi} = -1$, we can consider the set of Boolean matrices and the set of sign patterns as subclasses of the set of ray patterns. Table 1 shows the addition and the multiplication of 0 and rays.

Table 1: Addition and multiplication of 0 and rays

$+$	$e^{\mathrm{i}\theta_1}$	0	$\#$
$e^{\mathrm{i}\theta_2}$	$e^{\mathrm{i}\theta_1}$ if $e^{\mathrm{i}\theta_1} = e^{\mathrm{i}\theta_2}$ $\#$ if $e^{\mathrm{i}\theta_1} \neq e^{\mathrm{i}\theta_2}$	$e^{\mathrm{i}\theta_2}$	$\#$
0	$e^{\mathrm{i}\theta_1}$	0	$\#$
$\#$	$\#$	$\#$	$\#$

$\cdot$	$e^{\mathrm{i}\theta_1}$	0	$\#$
$e^{\mathrm{i}\theta_2}$	$e^{\mathrm{i}(\theta_1+\theta_2)}$	0	$\#$
0	0	0	0
$\#$	$\#$	0	$\#$

In Table 1, we denote by $\#$ any sum of rays where at least two of the rays are distinct, and we call $\#$ the ambiguous entry. The product of the $m \times p$ ray pattern $A = [a_{st}]$ and the $p \times n$ ray pattern $B = [b_{st}]$ is defined as usual; the (s,t) entry of AB is $\sum_{k=1}^{p} a_{sk}b_{kt}$. Note that the product of two ray patterns does not always yield a ray pattern, since some entries of the product can be $\#$.

We say that an $n \times n$ ray pattern A is *powerful* if for each positive integer k, the matrix A^k has no #. For a powerful ray pattern A, consider the sequence $A = A^1, A^2, A^3, \cdots$. If this sequence has repetitions, we say the ray pattern A is *periodic*. Let A^l be the first one that is repeated. Write $A^l = A^{l+p}$ with the minimal $p > 0$. Then l is called the *base* of A, and p the *period* of A. Denote the base of A by $l(A)$, and the period of A by $p(A)$. Note that if a powerful ray pattern A is periodic, then A^k is also periodic for each positive integer k.

The authors would like to point out that the definition of the periodicities of ray patterns in this paper is not general. Consider the following ray pattern

$$A = \begin{bmatrix} 0 & 1 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $A^4 = A^3$ but A^2 contains an ambiguous entry #. This example shows that it is possible to define the periodicities of ray patterns which are not powerful. We will refer to such matrices as *oscillatory* and they are a topic of future research. In [7], there is a general definition of the periodicities of sign patterns which are possibly not powerful, however in this thesis we restrict our attention to powerful matrices.

For a ray pattern $A = [a_{st}]$, we define the ray pattern $|A| = [a'_{st}]$ of A , where $a'_{st} = 1$ if $a_{st} \neq 0$ and $a'_{st} = 0$ if $a_{st} = 0$. Note that the entry 1 of the ray pattern $|A|$ is regarded as a ray, that is, $1 = e^{\mathrm{i}0}$. A square ray pattern D is called a *diagonal ray pattern*, if each diagonal entry of $|D|$ is 1 and other entries are 0. For ray patterns $A = [a_{st}]$ and $B = [b_{st}]$, we say that B is *ray diagonally similar* to A if there exists a diagonal ray pattern D satisfying $A = DBD^*$ and we write $A\tilde{B}$. We say that B is a *subpattern* of

A if $b_{st} = \delta_{st} a_{st}$ where δ_{st} is 1 or 0 for all s, t. If B is a ray subpattern of A, we write $B \preceq A$.

Note that each powerful sign pattern is periodic (See [7]). But for the ray pattern

$$A = e^{\mathrm{i}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

A is powerful but not periodic. In case of ray patterns, powerfulness does not guarantee periodicity. A ray ω is *periodic* if there exists a positive integer p satisfying the equation $\omega^p = 1$. And if ω is periodic, the smallest positive integer p satisfying $\omega^p = 1$ is called the *period* of ω, and is denoted by $p(\omega)$. In the previous example, we can see that A is not periodic since the ray e^{i} is not periodic.

The following is a basic proposition when we study powerful ray patterns.

Proposition 2.1 *(See Lemma 1.2 in [8]) The set of powerful ray patterns is closed under the following operations:*

(i) multiplication by any ray;

(ii) transposition;

*(iii) conjugate transposition (denoted by *);*

(iv) diagonal similarity;

(v) permutational similarity;

(vi) direct sum;

(vii) taking subpatterns.

Of particular interest in this thesis is the set of ray patterns A for which there exists a ray ω such that $A\tilde{\omega}|A|$, and we denote this set by S. In Theorem 3.3 it is shown that every irreducible powerful ray pattern is in S. We provide examples to show that this is not always the case for reducible powerful ray patterns.

Let $G = (V(G), E(G))$ be a digraph without multiple arcs. We define a *weighted digraph* $\mathcal{G}$ to be an ordered pair (G, w) where w is a function from $E(G)$ into the set of rays. We call the function w a *weight function* and the function value of an arc e in G, denoted by $w(e)$, the *weight* of e.

A *walk* in a digraph G is a sequence of edges from $E(G)$ of the form

$$(v_{k_1}, v_{k_2}), (v_{k_2}, v_{k_3}), \ldots, (v_{k_{l-1}}, v_{k_l}).$$

The number of edges in the walk is its *length* A *path* is a walk for which all of the vertices $v_{k_1}, v_{k_2}, \ldots, v_{k_l}$ are distinct. If $v_{k_1} = v_{k_t}$ we say that the walk is a *cycle*, and if all the vertices in a cycle (except the first and last) are distinct then we say the cycle is a *simple cycle*.

In Chapter 4.3, we consider semiwalks with forward and reversed edges and adopt the following more complicated notation in this case. We define a *semiwalk* W in a digraph to be a sequence of the form

$$W : v_{k_1}, e_{k_1}, v_{k_2}, e_{k_2}, \cdots, v_{k_l}, e_{k_l}, v_{k_{l+1}} \ (l \geq 1) \tag{2.1}$$

where each v_{k_i} is a vertex, each e_{k_i} is an arc of the form either $(v_{k_i}, v_{k_{i+1}})$ or $(v_{k_{i+1}}, v_{k_i})$. If there is no ambiguity, we abbreviate (2.1) to

$$W : v_{k_1} e_{k_1} v_{k_2} e_{k_2} \cdots v_{k_l} e_{k_l} v_{k_{l+1}} \ (l \geq 1) \tag{2.2}$$

Such l is called the *length* of the semiwalk and is denoted by $l(W)$. A semiwalk W is a *semicycle* if $v_{k_1} = v_{k_{l+1}}$. A semiwalk W is called a *semipath* if all the vertices in W

are different and is called a *simple semicycle* if all the vertices in W are different except $v_{k_1} = v_{k_{l+1}}$. If $e_{k_i} = (v_{k_{i+1}}, v_{k_i})$ and $v_{k_i} \neq v_{k_{i+1}}$, we call e_{k_i} a *reversed arc*; otherwise, we call e_{k_i} an *ordinary arc*. We define $a_+(W)$ and $a_-(W)$ to be the number of ordinary and the number of reversed arcs in W, respectively. A semiwalk of the form

$$v_{k_{l+1}} e_{k_l} v_{k_t} e_{k_{l-1}} \cdots v_{k_2} e_{k_1} v_{k_1} \ (l \geq 1)$$

is called the *reversed semiwalk* of W and is denoted by $\overline{W}$.

Note that a loop is an ordinary arc by definition. So for a vertex v, a semiwalk $W : v(v,v)v$ is a semicycle of length 1 and $\overline{W} = W$.

Suppose that $\mathcal{G} = (G, w)$ is a weighted digraph and G has a semiwalk W of the form (2.2). We define the sequence $\gamma(W; G, w)$ (or if there is no ambiguity, $\gamma(W; \mathcal{G})$)

$$\gamma(W; G, w) : \lambda_1, \lambda_2, \cdots, \lambda_l \text{ where } \lambda_i = (v_{k_i}, v_{k_{i+1}}; w(e_{k_i}))$$

for each i, and call it the *chain* of W with respect to w. The *reversed chain* $\overline{\gamma}(W; G, w)$ of W is the chain

$$\overline{\gamma}(W; G, w) : \overline{\lambda}_l, \overline{\lambda}_{l-1}, \cdots, \overline{\lambda}_1 \text{ where } \overline{\lambda}_i = (v_{k_{i+1}}, v_{k_i}; w(e_{k_i}))$$

for each i. So, by definition, $\overline{\gamma}(W; G, w) = \gamma(\overline{W}; G, w)$. If W is a semicycle or a cycle, we call $\gamma(W; G, w)$ a *semicyclic chain* or a *cyclic chain*, respectively. The *product* of $\gamma(W; G, w)$, denoted by $\wp(\gamma(W; G, w))$, is the ray defined by

$$\wp(\gamma(W; G, w)) = \left(\prod_{\substack{1 \leq i \leq l, \\ e_{k_i} \text{is ordinary}}} w(e_{k_i}) \right) \left(\prod_{\substack{1 \leq i \leq l, \\ e_{k_i} \text{is reversed}}} \overline{w(e_{k_i})} \right),$$

where the first or the second part is defined to be 1 if $a_+(W) = 0$ or $a_-(W) = 0$, respectively. Note that if W is a cycle of length 1, then $\wp(\overline{\gamma}(W; G, w)) = \wp(\gamma(W; G, w))$;

otherwise, $\wp(\overline{\gamma}(W;G,w)) = \wp(\gamma(\overline{W};G,w)) = \overline{\wp(\gamma(W;G,w))}$. And if W is a cycle, then $\wp(\gamma(W;G,w))$ is the product of all weights of arcs in W. Where no ambiguity arises we write $\wp(\gamma)$ for $\wp(\gamma(W;G,w))$

Let $A = [a_{ij}]$ be an $n \times n$ ray pattern. Then it is well-known that there exists a unique (up to graph isomorphisms) digraph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{(v_i, v_j) | a_{ij} \neq 0\}$. And we denote it by $G(A)$. Furthermore, if we consider not only the zero-nonzero pattern of A, but also the rays a_{ij}, we can determine a unique weight function w defined on $E(G)$ such that $w((v_i, v_j)) = a_{ij}$. So for a given square ray pattern A, there exists a unique weighted digraph (G, w), and we denote it by $\mathcal{G}(A)$. Throughout this thesis we will move fluidly between A and $\mathcal{G}(A)$.

Conversely, for a weighted digraph $\mathcal{G} = (G, w)$ with a vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$, there is a unique ray pattern $A = [a_{ij}]$ of order n, denoted by $A(\mathcal{G})$, such that

$$a_{ij} = \begin{cases} w((v_i, v_j)) & \text{if } (v_i, v_j) \in E(G), \\ 0 & \text{if } (v_i, v_j) \notin E(G). \end{cases}$$

Given an $n \times n$ matrix A, notice that the

$$(A^l)_{jk} = \sum_{W \in L(j,k,l)} wp(W)$$

where $L(j,k,l)$ is the set of all walks from v_j to v_k of length l in $\mathcal{G}(A)$

Let v_l and v_j be vertices in a graph G. If v_l has access to v_j and v_j has access to v_l, we say v_j and v_l *communicate*. The communication relation is an equivalence relation on the vertices of G, and thus we can partition V into equivalence classes which we will refer to as the *classes* of G.

A square matrix A is *reducible* if it is a 1×1 block of zeros or if there exists a permutation matrix P so that

$$P^T AP = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}.$$

where B and D are nonempty square matrices. The matrix A is *irreducible* if it is not reducible. Irreducibility is equivalent to the property that every two vertices v_i and v_j in $G(A)$ communicate. The classes of $G(A)$ correspond to the irreducible classes of A.

Let A be a reducible matrix. It is well know that A is permutationally similar to a matrix in Frobenius normal form, where each of the diagonal blocks is a square irreducible matrix or a 1×1 block of zeros:

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{mm} \end{bmatrix} \tag{2.3}$$

We define the *reduced graph of* A by $\mathcal{R}(A) = (V, E)$ where $V = \{ K \mid K$ is an irreducible class of $A \}$, and $E = \{ (K, L) \mid$ there is edge from a vertex $j \in K$ to a vertex $l \in L$ in $G(A) \}$. We will say that K is *nontrivial* if K is not the 1×1 block of zeros. We will say that a vertex K in $\mathcal{R}(A)$ is *initial* if it is not accessed by any other vertices in $\mathcal{R}(A)$ and we will say that it is *final* if it does not have access to any other vertex in $\mathcal{R}(A)$.

For an irreducible matrix A, the *index of imprimitivity* of A is the greatest common divisor of the lengths of the cycles in A, and is denoted by $k(A)$. If A is a zero matrix of order 1, $k(A)$ is undefined. For an irreducible matrix A, A is *primitive* if $k(A) = 1$ and A is *imprimitive* if $k(A) > 1$. It is well-known that for an irreducible matrix A with

$k(A) = k$, k is the greatest positive integer such that A is permutationally similar to matrix in block cyclic form

$$A = \begin{bmatrix} 0 & A_{1,2} & & & \\ & 0 & A_{2,3} & & \\ & & \ddots & \ddots & \\ & & & 0 & A_{k-1,k} \\ A_{k,1} & & & & 0 \end{bmatrix}, \tag{2.4}$$

where the zero diagonal blocks are square, and the nonzero blocks have no zero rows or zero columns (See [2]). When $k = 1$, A is in its own block cyclic form, and it will be understood that the block cyclic form (2.4) is $A_{1,1}$. For simplicity of notation, we may assume that A is already in block cyclic form (2.4).

Chapter 3

Irreducible Powerful Ray Patterns

3.1 Preliminary Work

In the paper [3], Cho, Kim, and I establish many interesting results which we will use later in this thesis and hence I have included it here as preliminary work. The work described in this section is also part of my MS Thesis under the supervision of Cho.

3.1.1 A Characterization of Irreducible Periodic Ray Patterns

In this section, we study irreducible ray patterns that are either powerful or periodic. Recall that by our definition, periodic ray patterns must be powerful. In the following, we denote by J the ray pattern each of whose entries is 1. We first consider irreducible powerful ray patterns.

Proposition 3.1 *(See Theorem 2.1 in [8]) Let A be an $n \times n$ ray pattern with no zero entries. Then A is powerful iff A is ray diagonally similar to $e^{i\theta}J$ for some $\theta \in R$.*

Proposition 3.2 *(See Theorem 3.5 in [8]) Every irreducible powerful ray pattern is a subpattern of a powerful ray pattern with no zero entries.*

From the above two propositions, we can obtain the following theorem which is rather simple but plays a major role throughout this thesis. Notice that this theorem implies that every irreducible powerful ray pattern is in S. We will see in Chapter 4 that this is not the case for reducible powerful ray patterns.

Theorem 3.3 *[3] Suppose that a ray pattern A is irreducible. Then A is powerful if and only if $A \in S$.*

Proof. 'If' part is trivial since a ray pattern $\omega|A|$ is powerful. Suppose an irreducible ray pattern A is powerful. Then, by Proposition 3.1, there exists a powerful ray pattern $\hat{A}$ with no zero entries such that A is a subpattern of $\hat{A}$. Moreover, by Proposition 3.2, there exists a diagonal ray pattern D such that $D\hat{A}D^* = \omega J$ for some ray ω. Since DAD^* is a subpattern of $D\hat{A}D^*$, each nonzero entry of DAD^* is ω. By noting that $|DAD^*| = |A|$, we have $DAD^* = \omega|A|$ and this completes the proof.

From Theorem 3.3, we can obtain an immediate corollary which is presented in [8].

Corollary 3.4 *(See Theorem 3.6 in [8]) Suppose that a ray pattern A is irreducible. Then A is powerful iff there exists a ray α such that αA is periodic.*

Proof. 'If' part is trivial. Suppose that an irreducible ray pattern A is powerful. By Theorem 3.3, A is ray diagonally similar to $\omega|A|$ for some ray ω. Let $\alpha = \omega^{-1}$. Then αA is ray diagonally similar to $|A|$, which is clearly an irreducible powerful sign pattern. Hence αA is periodic. □

For an irreducible powerful ray pattern A, we define the set

$$\Omega(A) = \{\omega \mid A \textit{ is ray diagonally similar to } \omega|A|\}.$$

From Theorem 3.3, $\Omega(A)$ is not empty. In Section 3.1.2, we consider the cardinality of $\Omega(A)$ and the geometric properties of the elements of $\Omega(A)$.

In [8], periodic ray patterns are characterized in terms of the powers. The following theorem characterizes irreducible periodic ray patterns in terms of the actual products of cycles. Note that diagonal similarities preserve the actual products of cycles.

Theorem 3.5 *[3] Suppose that an irreducible ray pattern A is powerful. Then A is periodic if and only if the actual product of each cycle in A is periodic.*

Proof. Since A is powerful, there exists a diagonal ray pattern D satisfying $DAD^* = \omega|A|$ for some ray ω.

Suppose A is periodic. Then ω is also periodic. Let γ be a cycle in A of length l. Since the actual products of cycles are invariant under diagonal similarities, the actual product $\wp(\gamma)$ of γ is ω^l. And ω^l is periodic because ω is periodic. We have just shown that the actual product of each cycle in A is periodic.

Next suppose that the actual product of each cycle in A is periodic. Let m_1 be the least common multiple of lengths of cycles in A and m_2 be the least common multiple of periodicities of actual products of cycles in A. Let $m = m_1 m_2$. Note that $DA^mD^* = (DAD^*)^m = \omega^m|A|^m = |A|^m$, since $\omega^m = 1$. Since $|A|$ is irreducible and m is a multiple of m_1, each diagonal entry of $|A|^m$ is 1. Hence each diagonal entry of DA^mD^* is 1. Note that diagonal similarities do not change the diagonal entries. Thus each diagonal entry of A^m is 1. Since $A^{2m} = A^mA^m$ and each diagonal entry of A^m is 1, A^m is a subpattern of A^{2m}. Similarly, A^{2m} is a subpattern of A^{3m} and so on. Since the order of A is finite, there exists a positive integer s such that $A^{sm} = A^{(s+1)m} = A^{sm+m}$. Therefore we have A is periodic. This completes the proof. □

In [7], the notion of *cyclically nonnegative sign patterns* was introduced. We extend this notion to ray patterns. A ray pattern A is *cyclically nonnegative* if the actual product of each cycle in A is 1. It is easy to see that an irreducible, cyclically nonnegative ray pattern is powerful.

Theorem 3.6 *[3] Suppose that a ray pattern A is irreducible. Then A is cyclically nonnegative iff A is ray diagonally similar to $\omega|A|$ for some ray ω satisfying $\omega^{k(A)} = 1$.*

Proof. Let $k(A) = k$ and $L(A) = \{l_1, l_2, \cdots, l_m\}$ be the set of lengths of the cycles in A. First assume that A is ray diagonally similar to $\omega|A|$ satisfying $\omega^k = 1$. Let γ be a cycle in A. Since the actual products of cycles are invariant under the diagonal similarities, the actual product $\wp(\gamma)$ of γ is $\omega^{l(\gamma)}$. Since $l(\gamma)$ is a multiple of k, $\omega^{l(\gamma)} = 1$. Thus A is cyclically nonnegative.

Now assume that A is cyclically nonnegative. Since A is irreducible and powerful, A is ray diagonally similar to $\omega|A|$ for some ray ω. We show that $\omega^k = 1$ as follows. Since k is the greatest common divisor of $L(A)$, we can take integers $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that $\sum_{s=1}^{m} \alpha_s l_s = k$. Then we have

$$\omega^k = (\omega^{l_1})^{\alpha_1} (\omega^{l_2})^{\alpha_2} \cdots (\omega^{l_m})^{\alpha_m}.$$

For each s, $(\omega^{l_s})^{\alpha_s} = (\wp(\gamma_s))^{\alpha_s}$ where γ_s is a cycle of length l_s. So we have

$$\omega^k = (\wp(\gamma_1))^{\alpha_1} (\wp(\gamma_2))^{\alpha_2} \cdots (\wp(\gamma_m))^{\alpha_m}.$$

By the assumption that A is cyclically nonnegative, we have $\wp(\gamma_s) = 1$ for each s. Therefore $\omega^k = 1$ and the theorem follows. □

In the following, we obtain the base and the period of an irreducible periodic ray pattern. By slightly modifying the proof of the well-known Lemma 1.2 in [7], we obtain the following proposition:

Proposition 3.7 *[3] Suppose that a ray pattern A is periodic. Then for positive integers m and k, $A^m = A^{m+k}$ iff $m \geq l(A)$ and $p(A)|k$.*

The following result is a generalization of Theorem 4.3 in [7].

Theorem 3.8 *[3] If an irreducible periodic ray pattern A is ray diagonally similar to $\omega|A|$, then $l(A) = l(|A|)$ and $p(A) = lcm\{p(\omega), p(|A|)\}$. Furthermore, if $k(A) = k$, then $p(A) = p(\omega^k)k$.*

Proof. By Theorem 3.3, without loss of generality, we may assume $A = \omega|A|$ since the base and the period are invariant under ray diagonal similarities. Let $p = lcm\{p(\omega), p(|A|)\}$. Then we have

$$\begin{aligned} A^{l(A)+p(A)} &= A^{l(A)}, \\ \omega^{l(A)+p(A)}|A|^{l(A)+p(A)} &= \omega^{l(A)}|A|^{l(A)}, \\ \omega^{p(A)}|A|^{l(A)+p(A)} &= |A|^{l(A)}. \end{aligned}$$

Since each nonzero entry of $|A|$ is 1, $\omega^{p(A)}$ must be 1 and hence $p(\omega)|p(A)$. From the last equality, we have $|A|^{l(A)+p(A)} = |A|^{l(A)}$. Thus we have $l(A) \geq l(|A|)$ and $p(|A|)|p(A)$ by Proposition 3.7. So $l(A) \geq l(|A|)$ and $p|p(A)$. Also we have

$$\begin{aligned} |A|^{l(|A|)+p} &= |A|^{l(|A|)}, \\ \omega^{l(|A|)+p}|A|^{l(|A|)+p} &= \omega^{l(|A|)+p}|A|^{l(|A|)}, \\ \omega^{l(|A|)+p}|A|^{l(|A|)+p} &= \omega^{l(|A|)}|A|^{l(|A|)}, \\ A^{l(|A|)+p} &= A^{l(|A|)}. \end{aligned}$$

It follows that $l(|A|) \geq l(A)$ and $p(A)|p$. Therefore we have $l(A) = l(|A|)$ and $p(A) = p = lcm\{p(\omega), p(|A|)\}$.

Note $p = lcm\{p(\omega), k\}$ since $p(|A|) = k$ (See [2]). Let $\alpha = p(\omega^k)$. We have $p(\omega)|\alpha k$ since $(\omega^k)^\alpha = \omega^{\alpha k} = 1$. Thus we have $p|\alpha k$. On the other hand, $\alpha|\frac{p}{k}$ because $(\omega^k)^{\frac{p}{k}} = 1$. Thus we have $\alpha k|p$. So $\alpha k = p$ and the theorem follows. ∎

Let A be an irreducible powerful sign pattern. Suppose that A is ray diagonally similar to $\omega|A|$. In the proof of Theorem 3.6, we see that $\omega^{k(A)}$ can be expressed as a product of actual products of cycles. Each actual product of cycles in A is 1 or -1 because A is a sign pattern. Thus, by Theorem 3.6, if A is cyclically nonnegative, then $\omega^{k(A)} = 1$ and if A has a negative cycle, then $\omega^{k(A)} = -1$. So, by Theorem 3.8, the following hold:

$$p(A) = \begin{cases} k & \text{if } A \text{ is cyclically nonnegative,} \\ 2k & \text{if } A \text{ has a negative cycle,} \end{cases}$$

and

$$l(A) = l(|A|).$$

Hence we can consider Theorem 3.8 is a generalization of Theorem 4.3 in [7].

Let A be an irreducible periodic ray pattern with $k(A) = k$. Suppose that A is already in block cyclic form (2.4). Then the Boolean matrix $|A|$ is

$$|A| = \begin{bmatrix} 0 & |A_{1,2}| & & & \\ & 0 & |A_{2,3}| & & \\ & & \ddots & \ddots & \\ & & & 0 & |A_{k-1,k}| \\ |A_{k,1}| & & & & 0 \end{bmatrix}.$$

It is well-known that $l(|A|)$ is the smallest positive integer l such that for all $s(1 \leq s \leq k)$, each entry of $|A_{s,s+1}||A_{s+1,s+2}|\cdots|A_{s+l-1,s+l}|$ is 1, where the indices are modulo k (See [7]). Each entry of $|A_{s,s+1}||A_{s+1,s+2}|\cdots|A_{s+l-1,s+l}|$ is 1 iff each entry of $A_{s,s+1}A_{s+1,s+2}\cdots A_{s+l-1,s+l}$ is not zero since A is powerful. From Theorem 3.8, we have $l(A) = l(|A|)$. Hence we can see that $l(A)$ is the smallest positive integer l such that for all $s(1 \leq s \leq k)$, each entry of $A_{s,s+1}A_{s+1,s+2}\cdots A_{s+l-1,s+l}$ is not zero, where the indices are modulo k. So we have shown the following:

Corollary 3.9 *[3] Suppose that A is an irreducible periodic ray pattern in block cyclic form (2.4) with $k(A) = k$. Then $l(A)$ is the smallest positive integer l such that for all $s(1 \leq s \leq k)$, each entry of $A_{s,s+1}A_{s+1,s+2}\cdots A_{s+l-1,s+l}$ is not zero, where the indices are modulo k.*

Now we characterize irreducible periodic ray patterns whose periods are p.

Theorem 3.10 *[3] Suppose that A is an irreducible ray pattern with $k(A) = k$. Then the following are equivalent:*

(i) A is periodic with period p;

(ii) k divides p and A is ray diagonally similar to $\omega|A|$ where $p(\omega^k) = p/k$.

Proof. Suppose that an irreducible ray pattern A is periodic with period p. Then A is ray diagonally similar to $\omega|A|$ for some ray ω by Theorem 3.3 and $p(A) = p(\omega^k)k$ by Theorem 3.8. Therefore k divides p and A is ray diagonally similar to $\omega|A|$ where $p(\omega^k) = p/k$.

Suppose that k divides p and A is ray diagonally similar to $\omega|A|$ where $p(\omega^k) = p/k$. Since ω is periodic, A is periodic. Since $k(|A|) = k$, $p = p(\omega|A|) = p(\omega^k)k = p$ by Theorem 3.8. Now the theorem follows. $\square$

Corollary 3.9 and Theorem 3.10 give an alternative proof for a result presented in [15].

Corollary 3.11 *(See Theorem 10 in [15]) Suppose that A is an irreducible ray pattern in block cyclic form (2.4) with $k(A) = k$. Then the following are equivalnet:*

(i) A is pattern p-potent for some positive integer p;

(ii) k divides p and A is ray diagonally similar to

$$\omega \begin{bmatrix} 0 & J_1 & & & \\ & 0 & J_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & J_{k-1} \\ J_k & & & & 0 \end{bmatrix},$$

where $p(\omega^k) = p/k$ and every J_s is a ray pattern each of whose entries is 1, and is the same size as the corresponding block $A_{s,s+1}$.

Proof. Let A be an irreducible ray pattern in block cyclic form (2.4) with $k(A) = k$. If A is a pattern p-potent ray pattern, then each entry of $A_{s,s+1}$ is not zero for every $s(1 \leq s \leq k)$ by Corollary 3.9. Thus we have

$$|A| = \begin{bmatrix} 0 & J_1 & & & \\ & 0 & J_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & J_{k-1} \\ J_k & & & & 0 \end{bmatrix},$$

where J_s is a ray pattern each of whose entries is 1, and is the same size as the corresponding block $A_{s,s+1}$. It follows that (i) implies (ii) by Theorem 3.10. It is easy to check that (ii) implies (i). This completes the proof. ∎

3.1.2 Cardinality of $\Omega(A)$ for an Irreducible Powerful Ray Pattern A

Let A be an irreducible powerful ray pattern. Recall that the set $\Omega(A)$ is

$$\Omega(A) = \{\omega \mid A \textit{ is ray diagonally similar to } \omega|A|\}.$$

By Theorem 3.3, $\Omega(A)$ is not empty. In this section, we study the cardinality of $\Omega(A)$ and the geometric property of the elements of $\Omega(A)$. We first consider a specific case.

Lemma 3.12 *[3] Suppose that a ray pattern A is in cyclic form*

$$A = \begin{bmatrix} 0 & \alpha_1 & & & \\ & 0 & \alpha_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & \alpha_{k-1} \\ \alpha_k & & & & 0 \end{bmatrix}$$

such that $\alpha_1 \alpha_2 \cdots \alpha_k = \alpha \neq 0$ and all other entries are 0. Then A is ray diagonally similar to

$$\beta \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

for each β satisfying $\beta^k = \alpha$.

Proof. First suppose that $\alpha = 1$. Let $\theta_s = arg(\alpha_s)$ for each $s(1 \leq s \leq k)$ and take $\theta \in R$. Take d_s where $arg(d_1) = \theta$ and $arg(d_s) = \theta + \sum_{j=1}^{s-1} \theta_j$ for $2 \leq s \leq k$, and let $D = diag\{d_1, d_2, \cdots, d_k\}$. Then, for $2 \leq s \leq k-1$, the argument $arg(d_s \alpha_s d_{s+1}^*)$ of $(s, s+1)$ entry of DAD^* is reduced to

$$\left(\theta + \sum_{j=1}^{s-1} \theta_j\right) + \theta_s - \left(\theta + \sum_{j=1}^{s} \theta_j\right) = 0 \pmod{2\pi}$$

. Also we have $arg(d_1 \alpha_1 d_2^*) = 0 \pmod{2\pi}$ and $arg(d_k \alpha_k d_1^*) = 0 \pmod{2\pi}$ since $arg(\alpha) = \sum_{j=1}^{k} \theta_j = 0 \pmod{2\pi}$. So each nonzero entry of DAD^* is 1. Therefore, if $\alpha = 1$, A is ray diagonally similar to $|A|$.

In the general case, suppose $\alpha \neq 0$. For each β satisfying $\beta^k = \alpha$, $\bar{\beta} A$ is ray diagonally similar to $|A|$. Thus, A is ray diagonally similar to $\beta|A|$ for each β satisfying $\beta^k = \alpha$ and this completes the proof. □

For a matrix A in the form

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix},$$

where each A_s is a square matrix for $1 \leq s \leq n$ and each of off-diagonal blocks is a zero matrix, we denote it by $\bigoplus_{s=1}^{n} A_s$.

Lemma 3.13 *[3] Suppose that an irreducible ray pattern A is ray diagonally similar to $\omega|A|$. Then A is ray diagonally similar to $\alpha|A|$ for each α satisfying $\alpha^{k(A)} = \omega^{k(A)}$.*

Proof. Let $k(A)=k$. Without loss of generality, we may assume that A is in block cyclic form

$$A=\omega\begin{bmatrix} 0 & |A_{1,2}| & & & \\ & 0 & |A_{2,3}| & & \\ & & \ddots & \ddots & \\ & & & 0 & |A_{k-1,k}| \\ |A_{k,1}| & & & & 0 \end{bmatrix}$$

and that its (s,s) diagonal block is of order n_s. By Lemma 3.12, for each α satisfying $\alpha^k=\omega^k$, there exists a diagonal ray pattern $D=diag\{d_1,d_2,\cdots,d_k\}$ such that

$$D\begin{bmatrix} 0 & \omega & & & \\ & 0 & \omega & & \\ & & \ddots & \ddots & \\ & & & 0 & \omega \\ \omega & & & & 0 \end{bmatrix}D^*=\alpha\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}.$$

Let $E=\bigoplus_{s=1}^{k} d_sI_s$, where I_s is a ray pattern of order n_s such that each of whose diagonal entries is 1 and each of whose off-diagonal entries is 0. Then the $(s,s+1)$ block of EAE^* is

$$d_sI_s(\omega|A_{s,s+1}|)\bar{d}_{s+1}I_{s+1}=\alpha|A_{s,s+1}|.$$

It follows that A is ray diagonally similar to $\alpha|A|$ for each α satisfying $\alpha^k=\omega^k$ and this completes the proof. ∎

Suppose that an irreducible ray pattern A is powerful. Then Lemma 3.13 implies that if A is ray diagonally similar to $\omega|A|$, then the set $\{x \mid x^{k(A)}=\omega^{k(A)}\}$ is a subset of $\Omega(A)$, hence $|\Omega(A)|\geq k(A)$.

Lemma 3.14 *[3] Suppose that an irreducible ray pattern A is powerful. If A is ray diagonally similar to both $\omega|A|$ and $\omega'|A|$, then $\omega^{k(A)} = (\omega')^{k(A)}$.*

Proof. Let $k(A) = k$ and $L(A) = \{l_1, l_2, \cdots, l_m\}$ be the set of lengths of the cycles in A. Assume that $\omega,\ \omega' \in \Omega(A)$. For each $s(1 \leq s \leq m)$, we can choose a cycle γ_s of length l_s. Note that for each s, $\omega^{l_s} = \wp(\gamma_s) = (\omega')^{l_s}$ because the actual products of cycles in A are invariant under diagonal similarities. Since k is the greatest common divisor of $L(A)$, there exist integers $\alpha_1, \alpha_2, \cdots,\ \alpha_m$ such that $\sum_{s=1}^{m} \alpha_s l_s = k$. We have

$$\omega^k = (\omega^{l_1})^{\alpha_1}(\omega^{l_2})^{\alpha_2}\cdots(\omega^{l_m})^{\alpha_m} = \{(\omega')^{l_1}\}^{\alpha_1}\{(\omega')^{l_2}\}^{\alpha_2}\cdots\{(\omega')^{l_m}\}^{\alpha_m} = (\omega')^k.$$

This completes the proof. □

It follows from Lemma 3.14 that $|\Omega(A)| \leq k(A)$. From Lemma 3.13 and Lemma 3.14, we can obtain the following theorem.

Theorem 3.15 *[3] Suppose that an irreducible ray pattern A is powerful. Then $|\Omega(A)| = k(A)$. Furthermore, we can label the elements of $\Omega(A)$ as $\omega_1, \omega_2, \cdots \omega_k$ such that $\omega_{s+1}/\omega_s = e^{\frac{2\pi}{k}\mathrm{i}}$ for $s = 1, 2, \cdots, k$, where $k(A) = k$ and $\omega_{k+1} = \omega_1$.*

Now we consider complex matrices. Let $A = [a_{st}]$ be a complex matrix. Each nonzero entry a_{st} of A can be decomposed into $amp(a_{st}) \cdot e^{\mathrm{i}\cdot arg(a_{st})}$, where $amp(a_{st})$ and $arg(a_{st})$ are the amplitude and the argument of a_{st} respectively. We define the complex matrix $arg(A) = [a'_{st}]$ to be

$$a'_{st} = \begin{cases} e^{\mathrm{i}\cdot arg(a_{st})} & \text{if } a_{st} \neq 0, \\ 0 & \text{if } a_{st} = 0. \end{cases}$$

Then by letting $amp(A) = [amp(a_{st})]$, A can be decomposed into $amp(A) \circ arg(A)$, where $\circ$ denotes the Hadamard product. Note that $amp(A)$ is a nonnegative matrix and $arg(A)$ can be regarded as a ray pattern. We denote the spectrum of A by $\sigma(A)$.

Theorem 3.16 *[3] Supppose that a complex matrix A is irreducible. If $arg(A)$ is cyclically nonnegative (that is, each actual product of cycles in $arg(A)$ is 1), then $\sigma(A) = \sigma(amp(A))$.*

Proof. Suppose that $arg(A)$ is cyclically nonnegative. By Theorem 3.6, there exists a unitary diagonal matrix D(in ray pattern sense, D can be considered as a diagonal ray pattern) such that $D\ (arg(A))\ D^* = amp(arg(A))$. Therefore, $DAD^* = D(amp(A) \circ arg(A))D^* = amp(A) \circ \{D\ (arg(A))\ D^*\} = amp(A) \circ amp(arg(A)) = amp(A)$. Thus we have $DAD^* = amp(A)$. Since the spectrum is invariant under the similarities, we have $\sigma(A) = \sigma(amp(A))$ and this completes the proof. □

The Perron-Frobenius Theorem is a well-known theorem about the spectrum of a nonnegative irreducible matrix (See [1]). Theorem 3.16 shows that an irreducible complex matrix A satisfies the Perron-Frobenius Theorem if $arg(A)$ is cyclically nonnegative. Based on this observation, we may regard Theorem 3.16 as a generalization of the Perron-Frobenius Theorem. For another generalization of the Perron-Frobenius Theorem, refer to [17].

3.2 The Minimum Upper Bound on the First Ambiguous Power of an Irreducible, Nonpowerful Ray or Sign Pattern

In this section we move from looking at ray patterns that are powerful, to those that are not powerful. In particular, we are interested in finding the first exponent t such that A^t contains an ambiguous entry. We conjecture that if A is an $n \times n$ irreducible ray pattern that is not powerful, then A^t contains an ambiguous entry for some positive integer t with $t \leq n^2 - 2n + 2$, and show that in all but one very special instance, this is the case. We also show that there is an $n \times n$ sign (and hence ray) pattern associated with the Wielandt graph, for which the first power that contains an ambiguous entry is the $n^2 - 2n + 2 - th$, and hence that the upper bound we give is, in fact, the minimum upper bound possible.

3.2.1 A Useful Lemma on Powers of Cycle Products

In this section we show that if A is an irreducible ray pattern with two simple cycles whose product weights raised to certain powers differ, then A^k has an ambiguous entry for some $k \leq n^2 - 2n + 2$. We begin with a short lemma that we will be used repeatedly in the proof of the main lemma of this section that following it.

Lemma 3.17 *Let A be an $n \times n$ irreducible ray pattern. If there exist cycles γ_1 and γ_2, with lengths l_1 and l_2, respectively, such that γ_1 and γ_2 share a common vertex, such that $l_1 + l_2 \leq 2n - 2$, and such that $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, where $m = lcm(l_1, l_2)$, then A^m has an ambiguous entry and $m < n^2 - 2n + 2$.*

Proof. Since $l_1 + l_2 \leq 2n-2$, we see that $m = \mathrm{lcm}(l_1, l_2) \leq l_1 l_2 \leq (n-1)^2 < n^2 - 2n + 2$. Let v_p be a common vertex between γ_1 and γ_2. For $j = 1, 2$, let β_j be the circuit through v_p obtained by following γ_j exactly $\frac{m}{l_j}$ times. Then each β_j has length m and weight $\wp(\gamma_j)^{\frac{m}{l_j}}$. Since $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, it follows that $(A^m)_{pp} = \#$. □

Lemma 3.18 *Let A be an $n \times n$ irreducible ray pattern. If there exist simple cycles γ_1 and γ_2 with lengths l_1 and l_2, respectively, such that $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, where $m = lcm(l_1, l_2)$, then A^k has an ambiguous entry for some $k \leq n^2 - 2n + 2$.*

Proof.

Case I: Suppose that γ_1 and γ_2 contain at least one common vertex; call it v_p.

By Lemma 3.17, we need only consider the case where $l_1 + l_2 > 2n - 2$. Since γ_1 and γ_2 are simple cycles on at most n vertices we see that $l_1 + l_2 \leq 2n$. We thus assume without loss of generality that $l_1 = n$, and that l_2 is either n or $n-1$. If $l_2 = n$, then there are two simple cycles of length n through v_p with different product weights, and hence, $(A^n)_{pp} = \#$. Thus we assume for the remainder of Case I that $l_2 = n - 1$, and hence, $m = n(n-1)$. Let H be the subgraph of $G(A)$ whose edges are precisely the edges common to γ_1 and γ_2.

Suppose first that H is a path α of length $n-2$. Let v_q be the first vertex in α and let v_r be the last vertex in α. Going around γ_1 exactly $n-1 = \frac{m}{l_1}$ times and around γ_2 exactly $n = \frac{m}{l_2}$ times, we see that $(A^{n(n-1)})_{rr} = \#$. By backtracking through the $n-2$ common vertices along α, we see that $(A^{n(n-1)-(n-2)})_{rq} = \#$. Note that $n(n-1) - (n-2) = n^2 - 2n + 2$.

Next we consider the case where H is not a path with length $n-2$. In this case, there are at least two disjoint edges in γ_1 that are not in γ_2. We can assume without loss of generality that the n-cycle γ_1 has edges labelled (v_j, v_{j+1}) for $j = 1, \ldots, n-1$ and edge (v_n, v_1). We also assume without loss of generality that that (v_1, v_2) and (v_h, v_{h+1}) are not edges in γ_2 for some h with $2 < h < n$ Since γ_2 has $n-1$ vertices, at least three of the vertices v_1, v_2, v_h, v_{h+1} are in γ_2; we can assume without loss of generality that v_1 and v_2 are vertices of γ_2. Let (v_1, v_k) be an edge in γ_2. Notice $k \neq 2$. Then γ_1 can be decomposed into three paths: $\alpha_1 = (v_1, v_2)$, α_2 from v_2 to v_k, and α_3 from v_k to v_1. Similarly γ_2 can be decomposed into three paths: $\beta_1 = (v_1, v_k)$, β_2 from v_k to v_2, and β_3 from v_2 to v_1. Then $\gamma_1\gamma_2 = \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3$. By following the same edges in a different order, we get three simple cycles, $\gamma_3 = \alpha_1\beta_3$, $\gamma_4 = \alpha_2\beta_2$, and $\gamma_5 = \alpha_3\beta_1$, with lengths l_3, l_4, and l_5, respectively.

Notice that $l_3 \leq 1 + n - 3 = n - 2$ and $l_5 \leq 1 + n - 2 = n - 1$. Let $m_j = \mathrm{lcm}(l_2, l_j)$ for $j = 3, 4, 5$. Since γ_2 has vertices in common with γ_3 and γ_5, by Lemma 3.17 we need to consider only the case where

$$\wp(\gamma_3)^{\frac{m_3}{l_3}} = \wp(\gamma_2)^{\frac{m_3}{l_2}} \text{ and } \wp(\gamma_5)^{\frac{m_4}{l_5}} = \wp(\gamma_2)^{\frac{m_5}{l_2}},$$

and hence

$$\wp(\gamma_3)^{l_2} = \wp(\gamma_2)^{l_3} \text{ and } \wp(\gamma_5)^{l_2} = \wp(\gamma_2)^{l_5},$$

If in addition,

$$\wp(\gamma_4)^{l_2} = \wp(\gamma_2)^{l_4},$$

then

$$\begin{aligned}\wp(\gamma_1)^{l_2}\wp(\gamma_2)^{l_2} &= \wp(\gamma_1\gamma_2)^{l_2}\\ &= \wp(\gamma_3\gamma_4\gamma_5)^{l_2}\\ &= \wp(\gamma_3)^{l_2}\wp(\gamma_4)^{l_2}\wp(\gamma_5)^{l_2}\\ &= \wp(\gamma_2)^{l_3+l_4+l_5}\\ &= \wp(\gamma_2)^{l_1+l_2}.\end{aligned}$$

Hence, $\wp(\gamma_1)^{l_2} = \wp(\gamma_2)^{l_1}$. Since $\gcd(l_1, l_2) = \gcd(n, n-1) = 1$, it follows that $m = \operatorname{lcm}(l_1, l_2) = l_1 l_2$, and hence

$$\wp(\gamma_1)^{\frac{m}{l_1}} = \wp(\gamma_2)^{\frac{m}{l_2}},$$

which contradicts one of our main assumptions. Thus for the remainder of Case I, we assume that

$$\wp(\gamma_4)^{l_2} \neq \wp(\gamma_2)^{l_4}.$$

By Lemma 3.17, we need only consider the case where $l_4 \geq n$. Since γ_4 does not go through v_1, it has at least n edges on at most $n-1$ vertices and hence is not a simple cycle. Decompose γ_4 into simple cycles $\gamma_6 \ldots \gamma_q$. Since γ_4 is made up of two paths α_2 and β_2, each γ_j for $j = 6, \ldots, q$ contains at least one vertex from β_2, and hence, from γ_2. Let $m_j = \operatorname{lcm}(l_2, l_j)$ for $j = 6, \ldots, q$. If

$$\wp(\gamma_j)^{\frac{m_j}{l_j}} = \wp(\gamma_2)^{\frac{m_j}{l_2}},$$

for $j = 6, \ldots, q$, then it is easy to see that

$$\wp(\gamma_4)^{l_2} = \wp(\gamma_2)^{l_4},$$

which is a contradiction. Thus there must exist $j \in \{6, \ldots, q\}$ such that $\wp(\gamma_j)^{\frac{m_j}{l_j}} \neq \wp(\gamma_2)^{\frac{m_j}{l_2}}$. Since γ_j is a simple cycle on at most $n-1$ vertices, $l_j \leq n-1$, and hence,

$l_2 + l_j \leq 2(n-1)$. By Lemma 3.17, there exists $k \leq n^2 - 2n + 2$ such that A^k contains an ambiguous entry.

Case II: Suppose that γ_1 and γ_2 have no vertices in common. Since A is irreducible, there is a path β_1 from some vertex v_p in γ_1 to some vertex v_q in γ_2 such that v_p is the only common vertex for γ_1 and β_1 and such that v_q is the only common vertex for γ_2 and β_1. Similarly there is a path β_2 from some vertex v_r in γ_2 to some vertex v_s in γ_1 such that v_r is the only common vertex for γ_2 and β_2 and such that v_s is the only common vertex for γ_1 and β_2. Note that β_1 and β_2 may have vertices and edges in common. Let β_3 be the path along γ_2 from v_q to v_r. Let β_4 be the path along γ_1 from v_s to v_p. (See Figure 1.) Then $\gamma_3 = \beta_1\beta_3\beta_2\beta_4$ is a circuit that has at least one vertex in common with each of γ_1 and γ_2.

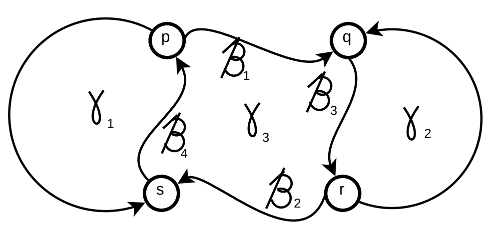

Figure 1: Connecting disjoint simple cycles

Let l_3 be the length of γ_3. Let $m_1 = \gcd(l_1, l_3)$ and $m_2 = \gcd(l_2, l_3)$.

We are now interested in the relationships between the three cycles γ_1, γ_2 and γ_3. Notice that in traversing γ_1, γ_2 and γ_3, we pass through each of the included vertices at most twice. So

$$l_1 + l_2 + l_3 \leq 2n,$$

and equality holds exactly when we pass through every vertex in G exactly twice while traversing $\gamma_1\gamma_2\gamma_3$.

Suppose first that

$$\wp(\gamma_2)^{\frac{m_2}{l_2}} \neq \wp(\gamma_3)^{\frac{m_2}{l_3}}.$$

Since $l_1 \geq 1$ it follows that $l_2 + l_3 \leq 2n - 1$. By Lemma 3.17, we need only look at the case where $l_2 + l_3 > 2n - 2$. Hence, we continue under the assumption that $l_2 + l_3 = 2n - 1$ and $l_1 = 1$. Write $l_2 = n - k$ where $k \geq 1$ and $l_3 = 2n - 1 - (n - k) = n + k - 1$. By our construction, the common edges between γ_2 and γ_3 form the path β_3. Since we must pass through every vertex in G exactly twice while traversing $\gamma_1\gamma_2\gamma_3$, it follows that β_3 must pass through every vertex of γ_2. That is, β_3 must cover all but one edge of γ_2, and hence it has length $n - k - 1$. Begin at v_r. By traversing γ_2 exactly $\frac{m_2}{l_2}$ times and by traversing γ_3 exactly $\frac{m_2}{l_3}$ times we see that $(A^{m_3})_{rr} = \#$. Backtracking along the path β_3 we get that $(A^w)_{rq} = \#$ where

$$\begin{aligned} w &= m_3 - (n - k - 1) \\ &\leq l_2 l_3 - (n - k - 1) \\ &= (n - k)(n + k - 1) - (n - k - 1) \\ &= n^2 - 2n + 2 - (k - 1)^2 \\ &\leq n^2 - 2n + 2 \end{aligned}$$

as desired.

Hence we assume that

$$\wp(\gamma_2)^{\frac{m_2}{l_2}} = \wp(\gamma_3)^{\frac{m_2}{l_3}},$$

and by an analogous argument, that

$$\wp(\gamma_1)^{\frac{m_1}{l_1}} = \wp(\gamma_3)^{\frac{m_1}{l_3}}.$$

Since the simple cycles γ_1 and γ_2 are disjoint, $l_1 + l_2 \leq n$. Without loss of generality, $l_2 \leq l_1$, and hence, when n is even, $l_2 \leq \frac{n}{2}$, and when n is odd, $l_2 \leq \frac{n-1}{2}$. By traversing γ_1 $\frac{m}{l_1}$ times and traversing γ_3 $\frac{m_2}{l_3}$ times, and by traversing γ_2 $\frac{m}{l_2}$ times and then traversing γ_3 $\frac{m_2}{l_2}$ times, we get two conflicting circuits of length $m + m_2$ through some vertex p common to the two circuits. Since $l_1 + l_2 + l_3 \leq 2n$, it follows that $l_1 + l_3 \leq 2n - l_2$. Note that $m = \operatorname{lcm}(l_1, l_2) \leq l_1 l_2$ and that $m_2 = \operatorname{lcm}(l_2, l_3) \leq l_2 l_3$. Then

$$m + m_2 \leq l_1 l_2 + l_2 l_3 = l_2(l_1 + l_3) \leq l_2\,(2n - l_2)$$

Since $f(x) = x(2n-x)$ is strictly increasing for $x \leq n$, $l_2\,(2n - l_2)$ is maximized at $l_2 = \frac{n}{2}$ when n is even, and at $l_2 = \frac{n-1}{2}$ when n is odd. Thus when n is even, $m + m_2 \leq \frac{3}{4}n^2$, and when n is odd, $m + m_2 \leq \frac{(n-1)(3n+1)}{4}$. Note that $\frac{3}{4}n^2 \leq n^2 - 2n + 2$ when $n \geq 4 + 2\sqrt{2} \approx 6.8$, so when n is even and $n \geq 8$, A^{m+m_2} has an ambiguous entry and $m + m_2 \leq n^2 - 2n + 2$. Since $\frac{(n-1)(3n+1)}{4} \leq n^2 - 2n + 2$ when $n \geq 5$, it follows that when $n \geq 5$ and odd, A^{m+m_2} has an ambiguous entry and $m + m_2 \leq n^2 - 2n + 2$. The remaining cases are $n = 2, 3, 4, 6$.

Note that for $n \geq 2$, $n \leq n^2 - 2n + 2$, for $n \geq 3$, $n + 2 \leq n^2 - 2n + 2$. We will construct conflicting walks in A^k for some $k \leq n$ when $n = 2$, and for some $k \leq n + 2$ when $n = 3, 4, 6$.

Suppose that the two disjoint simple cycles γ_1 and γ_2 for which $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$ holds are 1-cycles. Applying permutation similarity to A, we may assume that $\gamma_1 = (v_1, v_1)$ and $\gamma_2 = (v_n, v_n)$ with $\wp(\gamma_1) = a_{11}$, $\wp(\gamma_2) = a_{nn}$, and $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$ becomes

$a_{11} \neq a_{nn}$. Since A is irreducible, there is a path α of length ℓ with $\ell \leq n-1$ from v_1 to v_n. Then $\gamma_1\alpha$ and $\alpha\gamma_2$ are conflicting walks of length $\ell+1 \leq n$ from v_1 to v_n. (Note: this completes the $n=2$ case.)

Suppose that the two disjoint simple cycles γ_1 and γ_2 for which $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$ consist of a 1-cycle and a r-cycle for some $r \geq 2$. Applying permutation similarity to A, we may assume that $\gamma_1 = (v_1, v_1)$ and $\gamma_2 = (v_n, v_{n-r+1})(v_{n-r+1}, v_{n-r+2}) \cdots (v_{n-1}, v_n)$ with $\wp(\gamma_1) = a_{11}$,

$$\wp(\gamma_2) = a_{n,n-r+1} \prod_{j=2}^{r} a_{n-r+j-1,n-r+j} \ ,$$

and $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, which becomes $a_{11}^r \neq \wp(\gamma_2)$. Since A is irreducible, there is a path α of length ℓ with $\ell \leq n-r$ from v_1 to one of the vertices on γ_2 such that α only intersects γ_2 at a single vertex. Without loss of generality, that vertex is v_n. Then the walk obtained by traversing γ_1 r times followed by the path and and the walk α followed by traversing γ_2 are conflicting walks of length $\ell+r \leq n$ from v_1 to v_n. Further, $\ell+2 \leq n \leq n^2-2n+1$ for $n \geq 2$. (Note, with $r=2$, this completes the $n=3$ case.)

Suppose that the two disjoint simple cycles γ_1 and γ_2 for which $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$ comprises a pair of r-cycles for some $r \geq 2$. Applying permutation similarity to A, we may assume that $\gamma_1 = (v_1, v_2) \cdots (v_{r-1}, v_r)(v_r, v_1)$ and $\gamma_2 = (v_n, v_{n-r+1}) \cdots (v_{n-1}, v_n)$ with

$$\begin{aligned} \wp(\gamma_1) &= a_{r1} \textstyle\prod_{j=1}^{r-1} a_{j,j+1} \ , \\ \wp(\gamma_2) &= \textstyle\prod_{j=1}^{r} a_{n-r+j-1,n-r+j} \end{aligned}$$

and $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, which becomes $\wp(\gamma_1) \neq \wp(\gamma_2)$. Since A is irreducible, there is a path α of length ℓ with $\ell \leq n-2r+1$ from γ_1 to γ_2 such that α only intersects each of the simple cycles at a single vertex. Without loss of generality, those vertices are v_1 and v_n. Then $\gamma_1\alpha$ and $\alpha\gamma_2$ are conflicting walks of length $\ell+2r \leq n+1$ from v_1 to v_n.

(Note, with $r = 2$,this completes the $n = 4$ case.)

Suppose that the two disjoint simple cycles γ_1 and γ_2 for which $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$ comprises a 2-cycle and a 3-cycle. Applying permutation similarity to A, we may assume that $\gamma_1 = (v_1, v_2)(v_2, v_1)$ and $\gamma_2 = (v_n, v_{n-2})(v_{n-2}, v_{n-1})(v_{n-1}, v_n)$ with $\wp(\gamma_1) = a_{12}a_{21}$ and $\wp(\gamma_1) = a_{n,n-2}a_{n-2,n-1}a_{n-1,n}$, and $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, which becomes $\wp(\gamma_1)^3 \neq \wp(\gamma_2)^2$. Since A is irreducible, there is a path α of length ℓ with $\ell \leq n - 5 + 1 = n - 4$ from γ_1 to γ_2 such that α only intersects each of the simple cycles at a single vertex. Without loss of generality, those vertices are v_1 and v_n. Then the walk obtained by traversing γ_1 three times followed by α and the walk obtained by traversing α followed by traversing γ_2 twice are conflicting walks of length $\ell + 6 \leq n + 2$ from v_1 to v_n.

Finally, suppose that the two disjoint simple cycles γ_1 and γ_2 for which $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$ comprises a 2-cycle and a 4-cycle. Applying permutation similarity to A, we may assume that $\gamma_1 = (v_1, v_2)(v_2, v_1)$ and $\gamma_2 = (v_n, v_{n-3})(v_{n-3}, v_{n-2})\ (v_{n-2}, v_{n-1})(v_{n-1}, v_n)$ with $\wp(\gamma_1) = a_{12}a_{21}$ and $\wp(\gamma_1) = a_{n,n-3}a_{n-3,n-2}a_{n-2,n-1}a_{n-1,n}$ and $\wp(\gamma_1)^{\frac{m}{l_1}} \neq \wp(\gamma_2)^{\frac{m}{l_2}}$, which becomes $\wp(\gamma_1)^2 \neq \wp(\gamma_2)$. Since A is irreducible, there is a path α of length ℓ with $\ell \leq n - 6 + 1 = n - 5$ from γ_1 to γ_2 such that α only intersects each of the simple cycles at a single vertex. Without loss of generality, those vertices are v_1 and v_n. Then the walk obtained by traversing γ_1 twice followed by α and the walk obtained by traversing α followed by traversing γ_2 are conflicting walks of length $\ell + 4 \leq n - 1$ from v_1 to v_n. (Note that this completes the $n = 6$ case.) $\square$

3.2.2 The First Ambiguous Power

Lemma 3.19 *Let A be an irreducible ray pattern where the simple cycles $\gamma_1, \gamma_2, \ldots, \gamma_k$ are all the simple cycles in $\mathcal{G}(A)$. Let l_p be the length of γ_p, and $m_{pq} = lcm(l_p, l_q)$. Suppose that*

$$\wp(\gamma_p)^{\frac{m_{pq}}{l_q}} = \wp(\gamma_q)^{\frac{m_{pq}}{l_q}},$$

for all $1 \leq p, q \leq k$. Let $g = gcd(l_1, l_2, \ldots, l_k)$. If there exists $1 \leq j \leq k$ and $1 \leq p \leq k$, with $j \neq p$, such that $l_p = g(\frac{l_j}{g} s_p + u_p)$ where $gcd(u_p, \frac{l_j}{g}) = 1$, then A is powerful.

Proof. Without loss of generality assume $j = 1$ and $p = 2$. Choose ω such that $\wp(\gamma_1) = \omega^{l_1}$. Since A is powerful if and only if $\overline{\omega}A$ is powerful, we replace A by $\overline{\omega}A$, but continue to use the same notation. For each $1 \leq q \leq k$, write $\gcd(l_1, l_q) = gg_q$. Notice $m_{q1} g g_q = l_1 l_q$. Then

$$\wp(\gamma_q)^{\frac{m_{q1}}{l_q}} = \wp(\gamma_1)^{\frac{m_{q1}}{l_1}}$$

and thus

$$\wp(\gamma_q)^{\frac{l_1}{g g_{q1}}} = 1.$$

$$\wp(\gamma_q)^{\frac{l_1}{g}} = 1^{g_{q1}} = 1$$

$$\wp(\gamma_q)^{\frac{l_1}{g}} = 1.$$

Let $r = \frac{l_1}{g}$ and $\eta = \exp^{\frac{2\pi i}{r}}$. Then there exists $1 \leq t_q \leq r$ such that $\wp(\gamma_q) = \eta^{t_q}$.

Write $l_q = g(rs_q + u_q)$. Then m_{2q} divides $g(rs_q + u_q)(rs_2 + u_2)$ and hence

$$\wp(\gamma_q)^{\frac{m_{2q}}{l_q}} = \wp(\gamma_2)^{\frac{m_{2q}}{l_2}}$$

implies that

$$(\eta^{t_q})^{\frac{g(rs_q+u_q)(rs_2+u_2)}{l_q}} = (\eta^{t_2})^{\frac{g(rs_q+u_q)(rs_2+u_2)}{l_2}}$$

$$(\eta^{t_q})^{rs_2+u_2} = (\eta^{t_2})^{rs_q+u_q}$$

$$\eta^{t_q u_2} = \eta^{t_2 u_q}.$$

Thus $t_q u_2 \equiv t_2 u_q \bmod r$. Since, by assumption $\gcd(u_2, r) = 1$, we know that u_2 is invertible mod r. Thus

$$t_q \equiv u_q t_2 q_2^{-1} \bmod r. \tag{3.1}$$

Let α and β be any two cycles of the same length in $\mathcal{G}(\overline{\omega}A)$. Let m_q be the number of times the path α traverses γ_q, and n_q the number of times the path β traverses γ_q. Then

$$\sum_{q=1}^{k} m_q l_q = \sum_{q=1}^{k} n_q l_q$$

$$\sum_{q=1}^{k} m_q g(rs_q + u_q) = \sum_{q=1}^{k} n_q g(rs_q + u_q).$$

Dividing both sides by g, and collecting the terms with a factor of r on one side we get:

$$r\sum_{q=1}^{k} s_q(m_q - n_q) = \sum_{q=1}^{k} u_q(m_q - n_q).$$

and hence

$$\sum_{q=1}^{k} u_q(m_q - n_q) \equiv 0 \bmod r$$

so

$$\eta^{\sum_{q=1}^{k} u_q(m_q - n_q)} = 1 \text{ which implies that } \eta^{\sum_{q=1}^{k} u_q m_q} = \eta^{\sum_{q=1}^{k} u_q n_q}$$

Raising both side to the power $t_2 u_2^{-1}$ and substituting in for u_q from formula 3.1 we see that

$$\wp(\alpha) = \eta^{\sum_{q=1}^{k} t_q m_q} = \eta^{\sum_{q=1}^{k} t_q n_q} = \wp(\beta)$$

Thus we have shown that any two paths of the same length must have the same product weight in $\mathcal{G}(\overline{\omega}A)$ and hence in $\mathcal{G}(A)$.

Suppose A is not powerful. Then there exists a positive integer l and $1 \leq p, q \leq n$ such that $(A^l)_{pq} = \#$. This means that there are two paths (μ and ν) from p to q in $\mathcal{G}(A)$, both of length l, such that $\wp(\mu) \neq \wp(\nu)$. Since A is irreducible, there is a path υ from q to p. But then the cycles $\mu\upsilon$ and $\nu\upsilon$ from p to p have the same length but different product weights. This contradicts our claim that all cycles of the same length much have the same weight. Thus A must be powerful. □

Theorem 3.20 *Let A be an irreducible ray pattern that is not powerful. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be all the simple cycles in $\mathcal{G}(A)$. Let l_p be the length of γ_p. Let $g = gcd(l_1, l_2, \ldots, l_k)$. If there exists $1 \leq j \leq k$ and $1 \leq p \leq k$, with $j \neq p$, such that $l_p = g(\frac{l_j}{g} s_p + u_p)$ where $gcd(u_p, \frac{l_j}{g}) = 1$, then A^t contains an ambiguous entry for $t \leq n^2 - 2n + 2$.*

Proof. Follows from Lemma 3.18 and 3.19. □

At this point in time we are still working to determine whether our not upper bound on the exponent of the first ambiguous power still holds in the instance where, for all $1 \leq j \leq k$ and $1 \leq p \leq k$ with $j \neq p$, we have that $l_p = g(\frac{l_j}{g} s_p + u_p)$ with $\gcd(u_p, \frac{l_j}{g}) > 1$. This would happen if, for example, the simple cycles had lengths $6, 10$ and 15.

3.2.3 The Wielandt Graph

In this section we show that there is an $n \times n$ irreducible matrix A, for $n \geq 3$, that can be viewed as either a sign pattern or a ray pattern, such that the first power of A with an ambiguous entry is the $n^2 - 2n + 2 - th$ power. This establishes that $n^2 - 2n + 2$ cannot be replaced with a smaller power in our conjecture and Theorem 3.20.

The *Wielandt Graph* is the digraph $W = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ and

$$E = \{(v_i, v_{i+1}) | i = 1, \ldots, n-1\} \cup \{(v_n, v_1)\} \cup \{(v_{n-1}, v_1)\}.$$

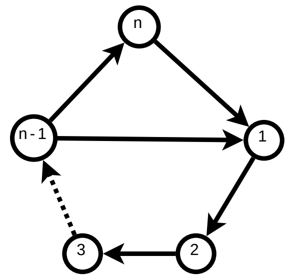

Figure 2: The Wielandt Graph

We consider the matrix $A = [a_{jk}]$ where

$$a_{jk} = \begin{cases} 1 = e^{i0} & \text{if k=j+1} \\ -1 = e^{i\pi} & \text{if } k = 1 \text{, and } \begin{cases} j = n & \text{if } n \text{ is even} \\ j = n-1 & \text{if } n \text{ is odd} \end{cases} \\ 1 = e^{i0} & \text{if } k = 1 \text{, and } \begin{cases} j = n & \text{if } n \text{ is odd} \\ j = n-1 & \text{if } n \text{ is even} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

Notice that $G(A) = W$, and A provides a weighting for for the edges of W. The graph W has exactly two simple cycles: an n-cycle γ_1 and an $n-1$-cycle γ_2, where

$$\wp(\gamma_1) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$

$$\wp(\gamma_2) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

Clearly, A is irreducible whether viewed as a sign pattern or as a ray pattern. If C is a cycle, then C must be obtained by traversing γ_1 r times for some $r \geq 0$ and traversing

γ_2 s times for some $s \geq 0$. Thus the length of C is $rn + s(n-1)$. If C_1 and C_2 are two distinct cycles of the same length, then $r_1 n + s_1(n-1) = r_2 n + s_2(n-1)$ with at least one of $r_1 \neq r_2$ and $s_1 \neq s_2$ holding. Further, if C_1 and C_2 are chosen so that there is no shorter pair of distinct cycles with a common length, then $\min(r_1, r_2) = 0$ and $\min(s_1, s_2) = 0$. Thus, without loss of generality, $r_1 n = s_1(n-1)$ with $r_1 s_1 \neq 0$. Since $\gcd(n, n-1) = 1$, the shortest pair occurs when $r_1 = n-1$ and $s_1 = n$. Thus for all j, $\left(A^k\right)_{jj}$ must be unambiguous for $k < n(n-1)$. Letting C_1 be the cycle obtained by traversing γ_1 $n-1$ times, $\wp(C_1) = \wp(\gamma_1)^{n-1}$. Letting C_2 be the cycle obtained by traversing γ_2 n times, $\wp(C_2) = \wp(\gamma_2)^n$. Note that $\wp(\gamma_1)^{n-1} = \wp(\gamma_1)$, and that $\wp(\gamma_2)^n = \wp(\gamma_2)$, so C_1 and C_2 are conflicting cycles of length $n(n-1)$. Consequently, the first occurrence of sharp in a diagonal entry of a power of A occurs for $A^{n(n-1)}$. Specifically, $\left(A^{n(n-1)}\right)_{n-1,n-1} = \#$. Since the two cycles share a common path of length $n-2$ from v_1 to v_{n-1}, it follows that $\left(A^{n(n-1)-n+2}\right)_{n-1,1} = \#$. Finally, observe that $n(n-1) - n + 2 = n^2 - 2n + 2$.

Suppose $(A^\ell)_{jk} = \#$. Then there are two walks β_1 and β_2 from v_j to v_k with length ℓ such that $\wp(\beta_1) = -\wp(\beta_2)$. Extend β_1 and β_2 to cycles C_1 and C_2 by adding the same shortest path γ from v_k to v_j of length h. Unless $j = 1$ and $k = n$, $h \leq n-2$. Note that C_1 and C_2 are distinct cycles in W with a common length, and hence their length must be at least $n(n-1)$. Unless $j = 1$ and $k = n$, the common length of β_1 and β_2 must be at least $n(n-1) - h \geq n(n-1) - (n-2) = n^2 - 2n + 2$. If $j = 1$ and $k = n$, then $h = n-1$ and the cycles C_1 and C_2 must traverse γ_1 because they contain v_n. Since both cycles are distinct but have the same length, it means that at least one must also traverse γ_2, without loss of generality, C_1 does. Then $r_1 n + s_1(n-1) = r_2 n + s_2(n-1)$ with r_1, r_2 and s_1 positive. From the argument given above, r_1 and s_1 positive implies that the common length of these cycles must exceed $n(n-1)$. Then the common length of β_1 and β_2 must exceed $n(n-1)-(n-1)=n^2-2n+2$.

Chapter 4

Reducible Powerful Matrices

In this Chapter we will look at properties of reducible powerful matrices.

Let A be a reducible matrix. It is well know that A is permutationally similar to a matrix in Frobenius normal form, where each of the diagonal blocks is a square irreducible matrix or a 1×1 block of zeros:

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{mm} \end{bmatrix} \tag{4.1}$$

Corollary 4.1 *Let A be a powerful ray pattern in Frobenius normal form as in (4.1). Then for $j = 1 \dots m$, there exists rays ω_j and diagonal ray patterns D_j such that*

$$D_j A_{jj} D_j^* = \omega_j |A_{jj}|$$

Proof. Follows from Theorem 3.3 and the observation that each diagonal block in the Frobenius normal form of A must itself be powerful. □

Let D be the diagonal ray pattern formed by taking the direct some of the diagonal ray patterns from Corollary 4.1. Let

$$DAD^* = \begin{bmatrix} \omega_{11}|A_{11}| & D_1A_{12}D_2^* & \dots & D_1A_{1m}D_m^* \\ 0 & \omega_{22}|A_{22}| & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \omega_{mm}|A_{mm}| \end{bmatrix} \tag{4.2}$$

We will refer to this as the *omega form* of A.

Throughout this chapter, let $\mathcal{G} = \mathcal{G}(A)$ and let $\mathcal{G}_i$ be the induced subgraph of $\mathcal{G}$ corresponding to the vertices and edges associated with the diagonal block A_{ii}.

4.1 Reducible Powerful Ray Patterns With Primitive Diagonal Blocks

We begin our study of reducible ray patterns by looking at the special case where A is a ray pattern such that all its irreducible classes are primitive.

Theorem 4.2 *Let A be a powerful $n \times n$ ray pattern in omega form (4.2). If each diagonal block of A is primitive, and $\mathcal{G}(A)$ is weakly-connected, then there exists a ray ω such that*

$$A = \omega \begin{bmatrix} |A_{11}| & \omega_{12}|A_{12}| & \dots & \omega_{1m}|A_{1m}| \\ 0 & |A_{22}| & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & |A_{mm}| \end{bmatrix} \tag{4.3}$$

Proof. Let A be a ray pattern in omega form (4.2) and assume that each diagonal block A_{ii} is primitive. Let (v_{i_1}, v_{j_1}) and (v_{i_2}, v_{j_2}) be arcs (possibly the same) from $\mathcal{G}_i$ to $\mathcal{G}_j$ in $\mathcal{G}$. Since A_{ii} and A_{jj} are primitive, there is an integer l_{ij} such that if $l \geq l_{ij}$, $\mathcal{G}_i$

and $\mathcal{G}_j$ have walks W_i and W_j of length l from v_{i_1} to v_{i_2} and from v_{j_1} to v_{j_2}, respectively. Let $W_1^{'}$ and $W_2^{'}$ be walks in $\mathcal{G}$ such that

$$W_1^{'} : W_1, (v_{i_2}, v_{j_2}), v_{j_2} \text{ and } W_2^{'} : v_{i_1}, (v_{i_1}, v_{j_1}), W_j.$$

Notice that $W_1^{'}$ and $W_2^{'}$ are walks of the same length from v_{i_1} to v_{j_2}. Since A is powerful we can conclude that

$$\wp(W_1^{'}) = \wp(W_2^{'}). \tag{4.4}$$

Hence in the case where $v_{i_1} = v_{i_2}$ and $v_{j_1} = v_{j_2}$, the equation (4.4) shows that $(\omega_{ii})^l = (\omega_{jj})^l$ for every l satisfying $l \geq l_{ij}$. So, in particular, we have $(\omega_{ii})^{l+1} = (\omega_{jj})^{l+1}$. Thus $\omega_{ii} = \omega_{jj}$. Since $\mathcal{G}$ is weakly connected, this implies that all the rays ω_{ii} in (4.2) are the equal. Let $\omega = \omega_{11} = \ldots = \omega_{mm}$.

Substituting $\omega_{ii} = \omega_{jj} = \omega$ into equation (4.4) we obtain $w((v_{i_1}, v_{j_1})) = w((v_{i_2}, v_{j_2}))$, and hence the nonzero entries in A_{ij} have the same value. □

Let A be a powerful ray pattern with primitive irreducible classes in the form (4.3), whose digraph $\mathcal{G}(A)$ is weakly connected. We now consider $\mathcal{R}(A)$, the reduced graph of A, and the corresponding matrix $R = R(A)$ where

$$r_{ij} = \begin{cases} \omega & \text{if } i = j \\ \omega\omega_{ij} & \text{if } i < j \\ 0 & \text{otherwise} \end{cases}$$

Notice $\mathcal{R}(A) = \mathcal{G}(R(A))$. Moreover, since $r_{ij} = 0$ if and only if $A_{ij} = 0$, we see that $\mathcal{R}(A)$ is weakly connected.

Lemma 4.3 *Let A be a powerful ray pattern such that each irreducible block is primitive,*

$\mathcal{G}(A)$ is weakly connected, and A is in the form (4.3). If A is powerful, then $R(A)$ is powerful.

Proof. We proceed by establishing the contrapositive. Suppose that $R = R(A)$ is not powerful. Then there exist two walks, W_1 and W_2, from irreducible class i to irreducible class j, in $\mathcal{R}(A)$, both of which have length k. Let p be any vertex associated with the irreducible class i in $\mathcal{G}(A)$. Let q be any vertex associated with the irreducible class j in $G(A)$. Suppose (p, q) is an edge in walk W_1 or W_2 with weight ω_{pq}. Then $A_{pq} \neq 0$, and since A_{pp} and A_{qq} are primitive with every edge having weight ω, there is a walk in $\mathcal{G}(A)$ from any vertex associated with the irreducible class p to any vertex associated with the irreducible class q, such that the weight of the walk is $\omega^{l-1}\omega_{pq}$, where l is the length of the walk. Let r be any vertex associated with the irreducible class i and s be any vertex associated with the irreducible class j. Then there is a walk W_3 from r to s such that $\wp(W_3) = \omega^{l_3-k}\wp(W_1)$, where l_3 is the length of W_3, and a walk W_4 from r to s such that $\wp(W_4) = \omega^{l_4-k}\wp(W_2)$. Since the irreducible class j is actually primitive, there exists a positive integer b such that there are cycles of length $b + t$, for all $t > 0$, from s to s, having weight ω^{b+t}. By adding cycles of the appropriate length from s to s, to W_3 and W_4, we end up with two walks from r to s in $\mathcal{G}(A)$, with the same length but different weights, and hence A is not powerful. □

Recall that if A is a ray subpattern of B, we write $A \preceq B$.

In the next few lemmas, we study matrices of the form

$$R = \begin{bmatrix} 1 & \omega_{12} & \cdots & \cdots & \omega_{1n} \\ 0 & 1 & \omega_{23} & \cdots & \omega_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \omega_{n-1,n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{4.5}$$

Lemma 4.4 *Let R be an upper triangular ray pattern in the form of (4.5). Then R is powerful if and only if $R^m = R^{m+1}$ for some $m \leq n-1$.*

Proof. Note that in ray pattern multiplication and addition,

$$R^k = (I+R)^k = I + R + R^2 + \cdots + R^k \tag{4.6}$$

(Only if part) If R is powerful, then R^k is a ray pattern for every $k \geq 0$. If $r_{ij}^k \neq 0$, then $r_{ij}^l = r_{ij}^k$ for all $l \geq k$. Since all paths in $\mathcal{G}(R)$ have length at most $n-1$, it follows that $R^{n-1} = R^n$.

(If part) If $R^m = R^{m+1}$ then $R^k = R^m$ for all $k \geq m$. In order for R^m to be well-defined, by (4.6) R^k is well-defined for $k \leq m$. □

Note that Lemma 4.4 shows that if R is powerful, R is periodic(with period 1) and the smallest such m is the base $l(R)$ of R.

Lemma 4.5 *Suppose that R is a powerful upper triangular matrix in the form (4.5). Then $R \in S$ if and only if $R^{l(R)} \in S$.*

Proof.

(Only if part) This is clearly true for all ray patterns.

(If part) Suppose not, that is, suppose that $R^{l(R)} \in S$ but $R \notin S$. Then $\mathcal{G}(R)$ has a semicycle with actual product not equal to 1, since R has all diagonal entries 1. But this would imply that $\mathcal{G}(R^{l(R)})$ has an alternating semicycle whose actual product is not equal to 1, contradiction. □

Example 4.6 *For every $n \geq 4$, there is a ray pattern (and sign pattern) R in form (4.5) such that R is powerful but $R \notin S$.*

Construction. If $n = 2k$, let A be the pattern with

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 1 & \text{if } i = 1 \text{ and } j = 3 \\ -1 & \text{if } i = 1 \text{ and } j = n \\ 1 & \text{if } i = 2q \text{ and } j = 2q+1 \text{ for } q = 1, 2, \ldots, k-1 \\ 1 & \text{if } i = 2q \text{ and } j = 2q+3 \text{ for } q = 1, 2, \ldots, k-2 \\ 1 & \text{if } i = n-2 \text{ and } j = n \\ 0 & \text{otherwise} \end{cases}$$

If $n = 2k+1$, let A be the pattern with

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 1 & \text{if } i = 1 \text{ and } j = 3 \\ -1 & \text{if } i = 1 \text{ and } j = n-1 \\ -1 & \text{if } i = 1 \text{ and } j = n \\ 1 & \text{if } i = 2q \text{ and } j = 2q+1 \text{ for } q = 1, 2, \ldots, k-1 \\ 1 & \text{if } i = 2q \text{ and } j = 2q+3 \text{ for } q = 1, 2, \ldots, k-1 \\ 1 & \text{if } i = n-1 \text{ and } j = n \\ 0 & \text{otherwise} \end{cases}$$

Notice in either case that $A^2 = A$ and hence A is powerful. Notice also that $\mathcal{G}(A)$ has a negative semicycle of length $2k+1$ with $k+1$ forward edges and k backward edges. We encourage the reader to come back to this example after having read Chapter 4.3, where Theorem 4.15 now shows that A is not in S.

These examples show that even simple reducible ray patterns can be powerful without being in S, and hence we devote Chapter 4.3 to establishing when a ray pattern is in S. In the next section, we look at reducible ray patterns whose irreducible blocks need not be primitive.

4.2 Reducible Powerful Matrices

We are now interested in looking at the more general case, where the diagonal blocks need not be primitive. We first look at an example to illustrate some of the differences in this case.

Example 4.7 Consider the following two matrices. Let $\omega_1 = e^{\frac{2\pi i}{6}}$, $\omega_2 = e^{\frac{2\pi i}{6}}$. and $\omega_3 = e^{\frac{2\pi i}{12}}$. Consider

$$A = \begin{bmatrix} 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_3 \\ 0 & 0 & 0 & 0 & 0 & \omega_3 & 0 & 0 & 0 \end{bmatrix}$$

Notice that A and B only differ in the $(3, 9)$-position, and in particular they have the same reduced graph. It is easy to check that A is powerful, while B is not.

Moreover, the matrix A shows us that although every reducible powerful ray pattern is similar to a ray pattern in omega form (4.2), primitivity is essential for the additional specifications in Theorem 4.2. In particular, if there were a diagonal matrix D and a ray ω, so that DAD^* was in the form of (4.3), then from the first diagonal block of A we would need $w^3 = -1$, from the second block that $\omega^2 = e^{\frac{4\pi i}{6}}$ and from the third block that $\omega^4 = e^{\frac{8\pi i}{12}}$, since the products of simple cycles in the graph of A are not changed by diagonally scaling A. But this implies that $-1 = \omega^3 = \omega\omega^2 = \omega e^{\frac{4\pi i}{6}}$, and $e^{\frac{8\pi i}{12}} = \omega^4 = \omega\omega^3 = -\omega$ and hence $e^{\frac{\pi i}{3}} = \omega = e^{\frac{5\pi i}{3}}$, a contradiction.

However, it is the case that we case that we can use Theorem 4.2 on selected powers of A in order to get a relationship between the values in each block.

Corollary 4.8 *Let A be a powerful ray pattern in omega form (4.2). If each A_{ii} contains at least one nonzero entry, then there exists a positive integer q and a ray ω such that*

A^q is permutationally diagonally similar to

$$\begin{bmatrix} \omega|A_{11}^q| & \omega_{12}|A_{12}^q| & \dots & \omega_{1k}|A_{1k}^q| \\ 0 & \omega|A_{22}^q| & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \omega|A_{kk}^q| \end{bmatrix} \tag{4.7}$$

(Note that the partitioning in this Frobenius normal form may differ from that in (4.2).)

Proof. Let c_i be the index of imprimitivity of A_{ii}. Let $q = \mathrm{lcm}(c_1, c_2, \dots, c_m)$. Then the diagonal blocks of A^q are primitive and hence our result follows from Theorem 4.2. □

Theorem 4.9 *Let A be an $n \times n$ ray pattern such that $\mathcal{G}(A)$ is weakly-connected. Suppose that every final vertex in $\mathcal{R}(A)$ is nontrivial. If A^s is well-defined for some positive integer s, then A^t is well-defined for each positive integer t such that $t < s$.*

Proof. Suppose A^s is well defined but A^t contains an ambiguous entry for some $t < s$. Then there are two vertices, v and w, and two paths P_1 and P_2, both of length t, from v to w, such that $\wp(P_1) \neq \wp(P_2)$. Since every final vertex in $\mathcal{R}(A)$ is nontrivial, we create a path of any length from w to some other vertex by following along a path until we have the desired length or we enter a final class in $\mathcal{R}(A)$. Since every final class is nontrivial it must contain a cycle and we can repeatedly transverse the cycle until the desired length is reached. Hence let P_3 be a path from w to some vertex u of length $s-t$. Then P_1P_3 and P_2P_3 are both paths from v to u of length s. But $\wp(P_1P_3) \neq \wp(P_2P_3)$ and this contradicts that A^s does not contain an ambiguous entry. □

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Example 4.10 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that A^2 is not well defined, however $A^k = 0$ for $k \geq 4$, hence if some of the final classes of $\mathcal{R}(A)$ are trivial, then A^s may be well defined even if A^t contains ambiguous entries for some $t < s$. We are interested in studying these types of patterns and they are discussed briefly in our concluding remarks.

Corollary 4.11 *Let A be an A be an $n \times n$ ray pattern such that $\mathcal{G}(A)$ is weakly-connected and every final vertex in $\mathcal{R}(A)$ is nontrivial. Let c_i be the index of imprimitivity of each irreducible block A_{ii} of A. Let $q = lcm(c_1, c_2, \ldots, c_m)$. Then A is powerful if and only if A^q is powerful.*

Notice that A^q has primitive blocks and hence we can use the results from Section 4.1 and work with A^q rather than A when working to determine whether or not A is powerful.

In the paper [9], Hall, Li and Stuart develop additional results for reducible powerful matrices and we encourage the interested reader to look at their article. We will now focus our attention on determining when a reducible powerful matrix is in S.

4.3 Powerful Ray Patterns and the Set S

In Section 3.1.1, we showed that a ray pattern A is irreducible, then A is powerful if and only if $A \in S$. In Example 4.6 we provide an example of a reducible powerful ray pattern that is not is S.

Suppose that $A = [a_{st}]$ is in S. Then there exist a ray ω and a diagonal ray pattern $D = diag\{d_1, d_2, \cdots d_n\}$ satisfying $DAD^* = \omega|A|$. Let $\hat{A} = [\hat{a}_{st}]$ such that

$$\hat{a}_{st} = \begin{cases} a_{st} & \text{if } a_{st} \neq 0, \\ \\ \bar{d}_s \omega d_t & \text{if } a_{st} = 0. \end{cases}$$

Clearly, $\hat{A}$ is irreducible and A is a subpattern of $\hat{A}$. Moreover $d_s \hat{a}_{st} \bar{d}_t = \omega$ for each s, t. So $D\hat{A}D^*$ is diagonally similar to ωJ. Hence A is a subpattern of an irreducible powerful ray pattern $\hat{A}$. Thus we have shown the following:

Proposition 4.12 *A ray pattern A is in S iff there exists an irreducible powerful ray pattern B such that A is a subpattern of B.*

Note that the "if part" of Proposition 4.12 is trivial.

In view of Proposition 4.12, a ray pattern A is in S iff we can extend A to an irreducible powerful ray pattern by replacing zero entries of A with some rays. The focus of the next two sections exploits this idea to study reducible ray patterns from the set S.

4.3.1 Characterization of S in Terms of Products of Chains

Lemma 4.13 *Let A_1 and A_2 be ray patterns such that $\mathcal{G}(A_1) = (G, w_1)$ and $\mathcal{G}(A_2) = (G, w_2)$. Suppose that W is a semicycle in G, and that γ_1 and γ_2 are the chains of W in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. If $A_1 \sim A_2$, then $\wp(\gamma_1) = \wp(\gamma_2)$.*

Proof. The product of the chain of a semicycle W is the product of products of the chains of simple semicycles in W. So we only need to consider the case that W is a simple semicycle. And without loss of generality, we may assume that W is a semicycle in the form of

$$W : v_1 e_1 v_2 e_2 \cdots v_l e_l v_{l+1} \ (l \geq 1)$$

where $v_{l+1} = v_1$. Clearly the result holds, if W or $\overline{W}$ is a cycle. And note that ray diagonal similarities preserve the assignments of loops. Thus we may assume that W is a semicycle which contains reversed arcs.

Then W has a vertex v_i such that $e_i = (v_i, v_{i+1})$ and $e_{i-1} = (v_i, v_{i-1})$ where the indices are modulo l. Let W' be the semicycle of the form

$$W' : v_i e_i v_{i+1} e_{i+2} \cdots v_{i-1} e_{i-1} v_i,$$

that is, vertices and arcs of W' are equal to those of W but the starting vertex is changed from v_1 to v_i. It is clear that $\wp(W) = \wp(W')$. Thus, again, without loss of generality, we may assume that $e_1 = (v_1, v_2)$ and $e_l = (v_1, v_l)$. Let P_1 be the longest path of forward arcs starting from v_1 in W. Since W has reversed arcs, the end vertex of P_1 is not v_1. Let $\overline{P}_2$ be the longest path which starts at the end vertex of P_1 in W. If end vertex of $\overline{P}_2$ is not v_1, similar to the case of P_1, we can take the longest path P_3 which starts at the end vertex of $\overline{P}_2$ in W. Note that the end vertex of P_3 is not v_1 since $e_1 = (v_1, v_2)$

and $e_l = (v_1, v_l)$. Again we can take the longest path $\overline{P}_4$ which starts at the end vertex of P_3 in W similar to the case of $\overline{P}_2$ and so on.

Then W is divided into an even number of semipaths $P_1, P_2, \cdots, P_{2m}$, where the even subscripted paths have only forward arcs and the odd subscripted paths have only reversed arcs. Let $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ be the chains of P_i in G_i for $i = 1, 2$, respectively. Suppose that the length of P_j is l_j for $j = 1, 2, \cdots, 2m$. Then

$$P_j : v_{j,1} e_{j,1} v_{j,2} v_{j,2} e_{j,2} v_{j,3} \cdots v_{j,l_j} e_{l_j} v_{j,l_j+1}.$$

Since $A_1 \sim A_2$, for $j = 1, 2, \cdots, 2m$ and $k = 1, 2, \cdots, l_j + 1$, there are rays $d_{j,k}$ such that if j is odd,

$$\gamma_j^{(2)} : (e_{j,1}; d_{j,1} w(e_{j,1}) \bar{d}_{j,2}), (e_{j,2}; d_{j,2} w(e_{j,2}) \bar{d}_{j,3}), \cdots, (e_{j,l_j}; d_{j,l_j} w(e_{j,l_j}) \bar{d}_{j,l_j+1})$$

and if j is even,

$$\gamma_j^{(2)} : (e_{j,1}; \bar{d}_{j,1} w(e_{j,1}) d_{j,2}), (e_{j,2}; \bar{d}_{j,2} w(e_{j,2}) d_{j,3}), \cdots, (e_{j,l_j)}; \bar{d}_{j,l_j} w(e_{j,l_j}) d_{j,l_j+1}).$$

Hence we have

$$\wp(\gamma_j^{(2)}) = d_{j,1} \wp(\gamma_j^{(1)}) \bar{d}_{j,l_j+1}$$

for each j. For $i = 1, 2$, the chain $\gamma_i : \gamma_1^{(i)}, \gamma_2^{(i)}, \cdots, \gamma_{2m}^{(i)}$ is the chain of W in G_i. Then we have

$$\wp(\gamma_2) = \prod_{j=1}^{2m} \wp\left(\gamma_j^{(2)}\right) = \left(\prod_{j=1}^{2m} \wp\left(\gamma_j^{(1)}\right)\right) \left(\prod_{j=1}^{2m} d_{j,1}\right) \left(\prod_{j=1}^{2m} \bar{d}_{j,l_j+1}\right) = \wp(\gamma_1)$$

since $d_{j+1,1} = d_{j,l_j+1}$ where the indices are modulo $2m$. This completes the proof. $\square$

By using Lemma 4.13, we can easily obtain a necessary condition for a ray pattern A to be in S.

Proposition 4.14 *Let A be a ray pattern of order n $(n \geq 2)$ and $\mathcal{G} = \mathcal{G}(A)$. If A is in S, then for each semicycle W in $\mathcal{G}$ with $a_+(W) = a_-(W)$, $\wp(\gamma(W;\mathcal{G})) = 1$.*

Proof. By Lemma 4.13, products of semicycles are invariant under ray diagonal similarities. Hence we may assume that $A = \omega|A|$ for some ray ω. Let W be a semicycle of length l in $G(A)$ with $a_+(W) = a_-(W)$. Then l is even and W contains exactly $\frac{l}{2}$ reversed arcs. Hence the product of the chain of W is $\omega^{\frac{l}{2}}(\bar{\omega})^{\frac{l}{2}} = 1$. This completes the proof. $\square$

Notice that Example 4.6 shows that this proposition is necessary but not sufficient.

Theorem 4.15 *Let A be a ray pattern of order n and ω be a ray. Suppose that $G(A)$ is weakly connected. Then $A \sim \omega|A|$ iff*

$$\wp(\gamma) = \omega^{a_+(\gamma) - a_-(\gamma)} \tag{4.8}$$

for each semicyclic chain γ in $G(A)$.

Proof. (Only If Part) Trivial by Lemma 4.13.

(If Part) Let $G(A) = G_0 = (V, E_0, w_0)$ where $V = \{v_1, v_2, \cdots, v_n\}$. By assumption, all loops in G_0 must have the assignment ω. If G_0 has vertices which are not on loops, we can attach a loop for each of such vertices and give assignment ω to each of new arcs. Let $G_1 = (V, E_1, w_1)$ be the resulting digraph such that

$$w_1(e) = \begin{cases} \omega_0(e) & \text{if } e \in E_0, \\ \omega & \text{if } e \in E_1 \setminus E_0. \end{cases}$$

Clearly, each semicycle W in G_1 satisfies (4.8).

Suppose that there exist distinct vertices v_{i_1}, v_{j_1} such that G_1 does not have the arc $e_1 = (v_{i_1}, v_{j_1})$. Since G_1 is weakly connected, there exists a semipath from v_{j_1} to v_{i_1} in G_1. Fix such a semipath and denote it by P_1. Define a diagraph G_2 to be $G_2 = (V, E_1 \cup \{e_1\}, w_2)$ such that

$$w_2(e) = \begin{cases} \omega_1(e) & \text{if } e \neq e_1, \\ \omega^{a_+(P_1)-a_-(P_1)+1}\wp(\overline{P}_1) & \text{if } e = e_1. \end{cases}$$

We show that each semicycle W in G_2 satisfies (4.8) as follows. If a semiwalk W in G_2 does not contain the arc e_1, W satisfies (4.8). Suppose that a semicycle W in G_2 contains e_1. Without loss of generality, we may assume that W is of the form $W : v_{i_1}(v_{i_1}, v_{j_1})P$ where P is a semipath from v_{j_1} to v_{i_1} in G_1. Note that the product of the chain of the semicycle $\overline{P}_1 P$ is

$$\wp(\overline{P}_1 P) = \wp(\overline{P}_1)\wp(P) = \omega^{a_-(P_1)+a_+(P)-a_+(P_1)-a_-(P)}.$$

By noting that $a_+(W) = a_+(P) + 1$ and $a_-(W) = a_-(P)$, we have

$$\begin{aligned} \wp(W) &= w(e)\wp(P) \\ &= \omega^{a_+(P_1)-a_-(P_1)+1}\wp(\overline{P}_1)\wp(P) \\ &= \omega^{a_+(P)-a_-(P)+1} \\ &= \omega^{a_+(W)-a_-(W)}. \end{aligned}$$

If there are distinct vertices v_{i_2} and v_{j_2} such that G_2 does not have the arc $e_2 = (v_{i_2}, v_{j_2})$, we can apply the same arguments of G_1 to G_2 and obtain the digraph $G_2 = (V, E_1 \cup \{e_1, e_2\}, w_2)$ such that each semicycle W in G_2 satisfies (4.8) and so on. Hence we can obtain a finite sequence of digraphs $G(A) = G_0, G_1, G_2, \cdots, G_m = G$ such that G has an arc for each pair of vertices and satisfies (4.8) for each semicycle.

Let $B = [b_{ij}]$ be the ray pattern which is associated with the digraph G. For each pair of i, j $(i < j)$, we can take two semicycles C_1 of length $j - i + 1$ and C_2 of length 2 in G

$$W_1 : v_i(v_i, v_{i+1})v_{i+1}(v_{i+1}, v_{i+2}) \cdots v_{j-1}(v_{j-1}, v_j)v_j(v_i, v_j)v_i,$$

$$W_2 : v_i(v_i, v_j)v_j(v_j, v_i)v_i.$$

Since W_1 and W_2 satisfy (4.8), we have

$$\wp(W_1) = b_{i,i+1}b_{i+1,i+2} \cdots b_{j-1,j}\overline{b}_{ij} = \omega^{j-i-1} \quad \text{and} \quad \wp(W_2) = b_{ij}b_{ji} = \omega^2.$$

Hence for each i, j $(i < j)$, we have

$$b_{ij} = \overline{\omega}^{j-i-1}b_{i,i+1}b_{i+1,i+2} \cdots b_{j-1,j} \quad \text{and} \quad b_{ji} = \omega^2\overline{b}_{ij}.$$

Let $D = \{d_1, d_2, \cdots, d_n\}$ be a diagonal ray pattern such that $d_1 = 1, d_{i+1} = \overline{\omega}d_i a_{i,i+1}$ $(1 \leq i \leq n-1)$. Then for each i, j $(i < j)$, the (i, j) entry of DBD^* is

$$\begin{aligned} d_i b_{ij}\overline{d}_j &= d_i \left(\overline{\omega}^{j-i-1}b_{i,i+1}b_{i+1,i+2} \cdots b_{j-1,j}\right)\overline{d}_j \\ &= \overline{\omega}^{j-i-1}\prod_{k=i}^{j-1}\left(d_k b_{k,k+1}\overline{d}_{k+1}\right) \\ &= \overline{\omega}^{j-i-1}\omega^{j-i} \\ &= \omega \end{aligned}$$

and the (j, i) entry of DAD^* is

$$d_j b_{ji}\overline{d}_i = d_j\left(\omega^2\overline{b}_{ij}\right)\overline{d}_i = \omega^2\overline{d}_i\overline{b}_{ij}d_j = \omega.$$

Note that each diagonal entry of B is ω and ray diagonal similarities preserve diagonal entries. So we have $DBD^* = \omega|B|$. Since A is a subpattern of B, we can conclude that $DAD^* = \omega|A|$. This completes the proof. $\square$

If we consider irreducible ray patterns, we can obtain much simpler characterization of S than Theorem 4.15.

Theorem 4.16 *Let A be an irreducible ray pattern. Then $A \sim \omega|A|$ iff*

$$\wp(\gamma) = \omega^{\ell(\gamma)} \tag{4.9}$$

for each cyclic chain γ in $G(A)$.

Proof. (If Part) Trivial by Theorem 4.15.

(Only If Part) Let W be a semicycle in G of the form

$$W : P_{11}\overline{Q}_{12}P_{22}\overline{Q}_{23}, \cdots \overline{Q}_{q-1,q}P_{qq}\overline{Q}_{q1}$$

where P_{ii} is a path from a vertex v_{k_i} to a vertex w_{k_i} and $Q_{i,i+1}$ is a path from a vertex $v_{k_{i+1}}$ to a vertex w_{k_i} for $i = 1, 2, \cdots, q$. Since G is strongly connected, there is a path $R_{i,i+1}$ from w_{k_i} to $v_{k_{i+1}}$ for each $i = 1, 2, \cdots, q$ with $R_{q,q+1} = R_{q1}$. Let

$$\ell_{ii} = \ell(P_{ii}), \quad \ell_{i,i+1} = \ell(Q_{i,i+1}), \quad \ell'_{i,i+1} = \ell(R_{i,i+1}),$$
$$\wp_{ii} = \wp(P_{ii}), \quad \wp_{i,i+1} = \wp(Q_{i,i+1}), \quad \wp'_{i,i+1} = \wp(R_{i,i+1}),$$

and W' be the closed walk of the form

$$W' : P_{11}R_{12}P_{22}R_{23}, \cdots, P_{qq}R_{q1}.$$

The length of the closed walk $Q_{i,i+1}R_{i,i+1}$ is $\ell_{i,i+1} + \ell'_{i,i+1}$ and the length of the closed walk W' is $\sum_{i=1}^{q}(\ell_{ii} + \ell'_{i,i+1})$. So we have

$$\wp(Q_{i,i+1}R_{i,i+1}) = \wp_{i,i+1}\wp'_{i,i+1} = \omega^{\ell_{i,i+1}+\ell'_{i,i+1}}$$

and

$$\wp(W') = \prod_{i=1}^{q} \wp_{ii}\wp'_{i,i+1} = \omega^{\sum_{i=1}^{q}(\ell_{ii}+\ell'_{i,i+1})},$$

From these two equations, we can have

$$\begin{aligned}
\wp(W) &= \left(\prod_{i=1}^{q} \wp_{ii}\right)\left(\prod_{i=1}^{q} \overline{\wp}_{i,i+1}\right) \\
&= \left(\prod_{i=1}^{q} \wp_{ii}\right)\left(\prod_{i=1}^{q} \wp'_{i,i+1}\right)\left(\prod_{i=1}^{q} \overline{\omega}^{\ell_{i,i+1}+\ell'_{i,i+1}}\right) \\
&= \omega^{\sum_{i=1}^{q}(\ell_{ii}+\ell'_{i,i+1})} \cdot \omega^{-\sum_{i=1}^{q}(\ell_{i,i+1}+\ell'_{i,i+1})} \\
&= \omega^{\sum_{i=1}^{q}\ell_{ii}-\sum_{i=1}^{q}\ell_{i,i+1}} \\
&= \omega^{a_+(W)-a_-(W)}.
\end{aligned}$$

Hence $A \sim \omega|A|$ from Theorem 4.15. This completes the proof. $\square$

In Theorem 4.15 and Theorem 4.16, if $\omega = 1$, a ray pattern A is ray diagonally similar to a Boolean matrix $|A|$. Hence in this case, many nice results about Boolean matrices (or nonnegative matrices) can be carried over to ray patterns (or complex matrices). In this point of view, next corollary is worth mentioning.

Corollary 4.17 *Let A be a ray pattern of order n ($n \geq 2$). Consider the following statements;*

(i) $A \sim |A|$;

(ii) $\wp(\gamma) = 1$ for each semicyclic chain γ in $G(A)$;

(iii) $\wp(\gamma) = 1$ for each cyclic chain γ in $G(A)$.

(i) and (ii) are always equivalent. If A is irreducible, (i), (ii) and (iii) are equivalent.

Let A be an irreducible ray pattern and $G(A) = G$. Denote the set of lengths of simple cycles in G by $L(G)$. For every $\ell \in L(G) = \{\ell_1, \ell_2, \cdots, \ell_m\}$, if A^ℓ is well-defined and all diagonal entries of A^ℓ are equal, we can define the multiset $\wp_{cyc}(G) = \{\wp_1, \wp_2, \cdots, \wp_m\}$

of products of cyclic chains G such that $\wp(C_i) = \wp_i$ for every simple cycle C_i with length ℓ_i.

Corollary 4.18 *Let A be an irreducible ray pattern of order n $(n \geq 2)$ and $G(A) = G$. Suppose that $L(G) = \{\ell_1, \ell_2, \cdots, \ell_m\}$ and $\sum_{s=1}^{m} p_s \ell_j = k(A)$ where each p_s is an integer. If there exists a ray ω such that for $1 \leq j \leq m$ and every simple cyclic chain γ of length ℓ_j in $G(A)$,*

$$\wp(\gamma) = \omega^{l_j}$$

then A is powerful and

$$\Omega(A) = \left\{ e^{\frac{\theta+2j\pi}{k(A)}\mathrm{i}} \middle| e^{\theta \mathrm{i}} = \prod_{s=1}^{m} \{\wp(\gamma_s)\}^{p_s} \text{ and } 1 \leq j \leq k \right\}.$$

Proof. Note that our condition implies that (4.9) holds for every cyclic chain γ in G. Hence by Theorem 4.16, A is powerful and we can find a ray ω such that $A \sim \omega|A|$. And

$$\omega^{k(A)} = \omega^{\sum_{s=1}^{m} p_s \ell_s} = \prod_{s=1}^{m} \{\wp(\gamma_s)\}^{p_s} = e^{\theta \mathrm{i}}.$$

So for each j $(1 \leq j \leq k(A))$, $\omega = e^{\frac{\theta+2j\pi}{k(A)}\mathrm{i}}$ is in $\Omega(A)$. However, $|\Omega(A)| = k(A)$ (See Theorem 3.15). Thus

$$\Omega(A) = \left\{ e^{\frac{\theta+2j\pi}{k(A)}\mathrm{i}} \middle| e^{\theta \mathrm{i}} = \prod_{s=1}^{m} \{\wp(\gamma_s)\}^{p_s} \text{ and } 1 \leq j \leq k(A) \right\}.$$

This completes the proof. □

4.3.2 Characterization of S in Terms of Powers

Now we consider the set S in terms of powers of ray patterns. To study this relation, we define a specific generalized ray pattern of a given ray pattern. For a ray pattern A and a ray α, we define a generalized ray pattern $A_{(\alpha)} = A + \alpha^2 A^*$.

Lemma 4.19 *Let A be a ray pattern and ω be a ray. Then*

(i) if $A \sim \omega|A|$, then $A_{(\omega)}$ is powerful;

(ii) if $G(A)$ is weakly-connected and $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2}$ is well-defined, then $A \sim \omega|A|$ or $-A \sim \omega|A|$.

Proof. (i) There exists a diagonal ray pattern D satisfying $DAD^* = \omega|A|$. We have

$$D(\omega^2 A^*)D^* = \omega^2(DAD^*)^* = \omega|A|^T.$$

Hence $DAD^* + D(\omega^2 A^*)D^*$ is well-defined. It follows that $A_{(\omega)} = A + \omega^2 A^*$ is well-defined. And $|A| + |A|^T = |A + \omega^2 A^*|$, thus we have $D(A + \omega^2 A^*)D^* = \omega(|A| + |A|^T) = \omega|A + \omega^2 A^*|$. Hence $A_{(\omega)} = A + \omega^2 A^*$ is powerful.

(ii) First note that $A_{(\omega)}$ is irreducible, so $\left(A_{(\omega)}\right)^m$ is well-defined for all m with $1 \leq m \leq l(|A_{(\omega)}|) + 2$.

We show that $A_{(\omega)} \sim \omega|A_{(\omega)}|$ or $-A_{(\omega)} \sim \omega|A_{(\omega)}|$. Then we can have $A \sim \omega|A|$ or $-A \sim \omega|A|$, since A is a subpattern of $A_{(\omega)}$. Note that

$$\left(A_{(\omega)}\right)^* = (A + \omega^2 A^*)^* = A^* + \overline{\omega}^2 A = \overline{\omega}^2 A_{(\omega)}.$$

Thus we have

$$\left(A_{(\omega)}\right)^2 = A_{(\omega)}\omega^2 \left(A_{(\omega)}\right)^* = \omega^2 A_{(\omega)} \left(A_{(\omega)}\right)^*.$$

Since $A_{(\omega)}$ is irreducible, each diagonal entry of $A_{(\omega)}\left(A_{(\omega)}\right)^*$ must be 1. It follows that $\omega^2 I$ is a subpattern of $\left(A_{(\omega)}\right)^2$. Hence $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2} = \left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)} \left(A_{(\omega)}\right)^2$ has $\omega^2 \left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)}$ as a subpattern. Since $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)}$ and $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2}$ have the same nonzero block pattern and each of nonzero blocks is entrywise nonzero, we have $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2} = \omega^2 \left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)}$. Multiplying both sides by $\bar{\omega}^{l(|A_{(\omega)}|)+2}$, we have

$\left(\bar{\omega}A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2} = \left(\bar{\omega}A_{(\omega)}\right)^{l(|A_{(\omega)}|)}$. It follows that $\bar{\omega}A_{(\omega)}$ is powerful, hence $A_{(\omega)}$ is powerful.

Now we can find a diagonal ray pattern D and a ray α such that

$$DA_{(\omega)}D^* = \alpha|A_{(\omega)}|.$$

Since A is a subpattern of $A_{(\omega)}$, we have

$$\begin{aligned} DA_{(\omega)}D^* &= D\left(A + \omega^2 A^*\right) D^* \\ &= \alpha|A| + \omega^2\overline{\alpha}|A^*|. \end{aligned}$$

From these two equations, we have $\alpha = \omega^2\overline{\alpha}$ or $\alpha = \pm\omega$. So we have $A_{(\omega)} \sim \omega|A_{(\omega)}|$ or $-A_{(\omega)} \sim \omega|A_{(\omega)}|$. Now the result follows. □

Theorem 4.20 *Let A be a ray pattern of order n ($n \geq 3$) and ω be a ray. Suppose that $G(A)$ is weakly connected. Consider the following statements;*

(i) $A \sim \omega|A|$ or $-A \sim \omega|A|$;

(ii) $A_{(\omega)}$ is powerful;

(iii) $\left(A_{(\omega)}\right)^{2n}$ is well-defined;

(iii)′ $\left(A_{(\omega)}\right)^{4n-6}$ is well-defined;

(iv) $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2}$ is well-defined.

If $G(A)$ has at least one odd semicycle, then (i), (ii), (iii) and (iv) are equivalent; otherwise, (i),(ii),(iii)′ and (iv) are equivalent.

Proof. (i) $\Rightarrow$ (ii): Note that $|-A| = |A|$ and $(-A)_{(\omega)} = -A_{(\omega)}$. So if $-A \sim \omega|A|$, then by Lemma 4.19, $-A_{(\omega)}$ is powerful, and hence $A_{(\omega)}$ is also powerful.

(ii) $\Rightarrow$ (iii): Immediate from the definition of powerfulness.

(iv) $\Rightarrow$ (i): It follows from Lemma 4.19.

Before we prove other implications, note that $A_{(\omega)}$ is irreducible with at least one cycle of length 2 since $G(A)$ is weakly-connected and $n \geq 3$. And note that $|A_{(\omega)}|$ is symmetric.

(iii) $\Rightarrow$ (iv): Now assume that $G(A)$ has at least one odd semicycle and $\left(A_{(\omega)}\right)^{2n}$ is well-defined. Since $A_{(\omega)}$ has at least one odd cycle, $|A_{(\omega)}|$ is a primitive Boolean matrix. Since $|A_{(\omega)}|$ is symmetric, we have

$$l(|A_{(\omega)}|) + 2 \leq 2(n-1) + 2 = 2n$$

(See [2]). Since $A_{(\omega)}$ is irreducible, $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2}$ is well-defined.

(iii)$'$ $\Rightarrow$ (iv): Assume that $G(A)$ has no odd semicycles and $\left(A_{(\omega)}\right)^{4n-6}$ is well-defined. Then $k(A_{(\omega)}) = k(|A_{(\omega)}|) = 2$. So without loss of generality, we may assume that $|A_{(\omega)}|$ is in the cyclic form

$$|A_{(\omega)}| = \begin{bmatrix} O & B \\ C & O \end{bmatrix}.$$

The diagonal blocks BC and CB of $\left|A_{(\omega)}\right|^2$ are primitive and both of them have order at least 1. Since $\left|A_{(\omega)}\right|^2$ is symmetric,

$$l(|A_{(\omega)}|^2) \leq 2(n-1) - 2 = 2n - 4.$$

And clearly we have

$$l(|A_{(\omega)}|) \leq 2l(|A_{(\omega)}|^2).$$

From these two inequalities, we finally have

$$l(|A_{(\omega)}|) + 2 \leq 2l(|A_{(\omega)}|^2) + 2 \leq 2(2n-4) + 2 = 4n - 6.$$

Since $A_{(\omega)}$ is irreducible, $\left(A_{(\omega)}\right)^{l(|A_{(\omega)}|)+2}$ is well-defined.

This completes the proof. □

We close this chapter by reconsidering Theorem 4.15 and Theorem 4.20. To apply Theorem 4.15, we need to check every semicycle and have to solve a system of equations. Theorem 4.15 may be more useful in showing that a ray pattern is not in S than showing that a ray pattern is in S. However, a real profit of Theorem 4.15 is that a semicyclic chain γ with $a_+(\gamma) - a_-(\gamma) \neq 0$ gives us possible ω such that $A \sim \omega|A|$. On the other hand, Theorem 4.20 tells us no informations on the ray ω in the statement. Theorem 4.20 is applied well to a ray pattern if we have informations on the ray ω. From these two observations, we can get the following algorithm for checking a ray pattern to be in S.

(Algorithm checking a ray pattern to be in S)

For a given ray pattern A of order $n \geq 3$ such that $G(A)$ is weakly-connected,

(i) Find a semicyclic chain γ in $G(A)$ with $a_+(\gamma) - a_-(\gamma) \neq 0$;

(ii) Solve $\wp(\gamma) = \omega^{a_+(\gamma)-a_-(\gamma)}$ for ω;

(iii) For rays ω obtained from (ii),

check $(A_{(\omega)})^{2n}$ is well-defined if $G(A)$ has a odd semicycle;

check $(A_{(\omega)})^{4n-6}$ is well-defined otherwise.

Step (i) and (ii) depend on Theorem 4.15, and step (iii) depends on Theorem 4.20. Note that $4n-6 \geq 2n$ if $n \geq 3$. Hence $(A_{(\omega)})^{2n}$ is well-defined if $(A_{(\omega)})^{4n-6}$ is well-defined since $A_{(\omega)}$ is irreducible. So instead of applying step (iii), we can simply compute $(A_{(\omega)})^{4n-6}$ without checking the existence of odd semicycles in $G(A)$. This alternative may be useful when n is a very large number. If n is very large $4n-6 \approx 4n$. So to compute $(A_{(\omega)})^{4n-6}$ and $(A_{(\omega)})^{2n}$, we need approximately $\log_2 4n$ and $\log_2 2n$ multiplications, respectively. The difference is $\log_2 4n - \log_2 2n = 1$. So unless we can find an odd semicycle easily, we can simply check $4n-6$-th power.

If every semicyclic chain γ satisfies $a_+(\gamma) - a_-(\gamma) = 0$, there are two possible cases. If there is a semicyclic chain whose product is not 1, then A is not in S by Theorem 4.15. If all products of semicyclic chains are 1, then $A \sim \omega|A|$ for an arbitrary ray ω again by Theorem 4.15.

Chapter 5

Concluding Remarks

We have several characterizations on irreducible powerful ray patterns. In general, however, characterizing powerful ray patterns is still open. In this paper, as a partial answer to this question, we have established several interesting results on the set S. In future work, we want to explore the properties of reducible ray patterns which are not powerful instead of studying powerful ray patterns characterization problem.

For a given irreducible ray pattern A, if A is not powerful then there exists an smallest integer m such that A^n is not well-defined for all n satisfying $n \geq m$. But this fact heavily depends on the irreducibility of a given ray pattern. Let's consider some examples.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A behaves exactly like irreducible non-powerful ray patterns, that is, A is not powerful and A^n in not well-defined for all $n \geq 2$. However, we can check that B^2 is not well-defined but B^n where $n \geq 3$ is well-defined. Also we can check that for all $n \geq 1$, C^{2n} is well-defined but C^{2n+1} is not. B shows that there is a ray pattern which is not powerful but "eventually" powerful and C is an example of ray pattern which "oscillates" between well-definedness and unwell-definedness. By classifying these three classes, we can approach the characterization of reducible powerful ray patterns.

In addition, for the class of ray patterns which contains B, it is interesting to find the smallest integer n such that the n-th or higher power is well-defined. And for a given increasing sequence $\{a_n\}$ of positive integers, considering if there is a ray pattern such that only a_n-th power is well-defined (or not well-defined) might be interesting.

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