SPECTRALLY ARBITRARY ZERO-NONZERO PATTERNS

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I certify that I have read this thesis and certify that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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Abstract

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This thesis establishes a complete list of all 3×3 and 4×4 complex spectrally arbitrary zero-nonzero patterns. Highlighted in this list are important examples of irreducible complex spectrally arbitrary zero-nonzero patterns which fail to satisfy the Nilpotent-Jacobian condition. Examples of complex spectrally arbitrary zero-nonzero patterns whose corresponding directed graph does not contain a two-cycle are illustrated in Chapter 3. In Chapter 4 the minimum number of nonzero entries contained in an irreducible zero-nonzero pattern that guarantees the pattern is spectrally arbitrary is determined. Illustrated in Chapter 5 is the reduction of this number of nonzero entries contained in a irreducible zero-nonzero pattern $\mathcal{A}$, when exactly one transversal is contained in an irreducible subpattern of $\mathcal{A}$. Lastly, future work in the area of spectrally arbitrary patterns is described in Chapter 6.

I would like to take this opportunity to thank my husband, Jason, for the many years of support and love that he has given me. Without him, I do not see how this thesis would be possible. I would like to thank my advisor, Judith McDonald, for all her patience and help throughout the past five years. Without her guidance, I might have been forever lost in the darkness of mathematics research. I would like to thank my many office mates, Nate Moyer, Corby Harwood, and Li Zhu, for their patience and help throughout the past five years. I would like to thank the support staff of the Mathematics department, for their willingness to always help a graduate student in need. Lastly, I would like to thank my parents for instilling in my soul the will to work hard and never give up.

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Chapter 1

Introduction

In this thesis we study the spectra of zero-nonzero patterns. Spectra of matrices with free entries were studied by Friedland in [8] and Farahat and Ledermann in [7]. In these papers free entries could be zero or nonzero, while in this thesis we require nonzero free entries. These papers along with others motivated many mathematicians to study the spectra of matrices with free nonzero entries.

Over the past decade, many results have been published concerning spectrally arbitrary patterns. Classification of families of spectrally arbitrary patterns were studied by Drew, Johnson, Olesky, and van den Driessche in [6]. Spectra of sign patterns [1, 2, 6, 10, 11, 12, 13, 17] and real zero-nonzero patterns [3, 4, 5, 9] have received considerable attention. Some study [14] has been accomplished for spectra of ray patterns. In this thesis we provide interesting examples illustrating fundamental differences between complex zero-nonzero patterns and real zero-nonzero patterns.

In [6] sufficient criterion for proving a pattern is spectrally arbitrary was discovered. Satisfying the Nilpotent-Jacobian condition is currently the most accessible method of proof for irreducible spectrally arbitrary zero-nonzero patterns. It is unknown if satisfying the Nilpotent-Jacobian condition is a necessary criterion. In Section 3.1 we discover complex spectrally arbitrary zero-nonzero patterns which do not satisfy the

Nilpotent-Jacobian condition. These patterns may be found in Appendix A: second pattern in the first row, and Appendix B: first pattern in the first row.

In [2] the authors show that there must exist at least one two-cycle in the directed graph of a real spectrally arbitrary pattern. In Section 3.2 we provide examples of complex spectrally arbitrary zero-nonzero patterns whose directed graph does not contain a two-cycle. These patterns may be found in Appendix A: third, fourth, and fifth pattern in the first row and first pattern in the second row.

We provide a complete list of all 3×3 (Section 3.4) and 4×4 (Section 3.5) complex spectrally arbitrary patterns in Chapter 3. It should be noted that up to equivalence, we found one 3×3 and seven 4×4 irreducible complex minimally spectrally arbitrary zero-nonzero patterns.

In Chapter 4 we prove that the minimum number of nonzero entries which guarantees an irreducible zero-nonzero pattern is spectrally arbitrary is $n^2 - 2n + 3$. We consider two main cases: the directed graph corresponding to the pattern contains an n-cycle and the directed graph corresponding to the pattern contains no n-cycle.

Lastly we display the reduction of this number of nonzero entries to $\frac{n(n+1)}{2} + 1$ for an irreducible zero-nonzero pattern $\mathcal{A}$, when exactly one transversal is contained in an irreducible subpattern of $\mathcal{A}$. It should be noted that the results in Chapters 4 and 5 hold for both real and complex zero-nonzero patterns.

Chapter 2

Definitions, notation and conventions

2.1 Patterns

A *zero-nonzero pattern* is a square matrix, $\mathcal{A}$, with entries in $\{0, *\}$, where $*$ represents a nonzero entry. We use the notation $a_{i,j}$ (resp. $b_{i,j}$), to distinguish between nonzero entries in $\mathcal{A}$ (resp. $\mathcal{B}$). A *complex zero-nonzero pattern* is a zero-nonzero pattern with $* \in \mathbb{C}\backslash\{0\}$. Similarly, a *real zero-nonzero pattern* is a zero-nonzero pattern with $* \in \mathbb{R}\backslash\{0\}$. A zero-nonzero pattern $\mathcal{B}$ is a *superpattern* of a zero-nonzero pattern $\mathcal{A}$ if $b_{i,j} = a_{i,j}$ whenever $a_{i,j} \neq 0$. A zero-nonzero pattern $\mathcal{B}$ is a *subpattern* of a zero-nonzero pattern $\mathcal{A}$ if $b_{i,j} = a_{i,j}$ whenever $b_{i,j} \neq 0$.

Example 2.1.1 $\mathcal{B} = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & * & * \end{bmatrix}$ *is a superpattern of* $\mathcal{A} = \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$.

$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ 0 & * & 0 \end{bmatrix}$ *is a subpattern of* $\mathcal{A} = \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$.

A zero-nonzero pattern $\mathcal{B}$ is a *proper subpattern* (resp. superpattern) of $\mathcal{A}$, if $\mathcal{B}$ is a subpattern (resp. superpattern) of $\mathcal{A}$ and $\mathcal{B} \neq \mathcal{A}$. We use $\mathcal{A}^-$ to denote a subpattern of

$\mathcal{A}$ where exactly one nonzero entry in $\mathcal{A}$ is a zero entry in $\mathcal{A}^-$. We use $\mathcal{A}^+$ to denote a subpattern of $\mathcal{A}$ where exactly one zero entry in $\mathcal{A}$ is a nonzero entry in $\mathcal{A}^+$. A *realization* of a complex (resp. real) zero-nonzero pattern $\mathcal{A}$ is a complex (resp. real) matrix R such that $r_{i,j} = 0$ if and only if $a_{i,j} = 0$; we write $R \in \mathcal{A}$.

Example 2.1.2 $R = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ .65 & \pi & 0 \end{bmatrix}$ *is a realization of* $\mathcal{A} = \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$.

An $n \times n$ complex (resp. real) zero-nonzero pattern $\mathcal{A}$ is *spectrally arbitrary* if for each complex (resp. real) monic polynomial $r(t)$ of degree n, there exists an $R \in \mathcal{A}$ where the characteristic polynomial of R is $p_R(t) = r(t)$. A zero-nonzero pattern $\mathcal{A}$ is *minimally spectrally arbitrary* if it is spectrally arbitrary and every proper subpattern of $\mathcal{A}$ is not spectrally arbitrary.

In this thesis, two zero-nonzero patterns, $\mathcal{A}$ and $\mathcal{B}$, are *equivalent* if there exists a permutation matrix P so that $\mathcal{A} = P\mathcal{B}P^T$ or $\mathcal{A} = P\mathcal{B}^T P^T$; we write $\mathcal{A} \sim \mathcal{B}$. Notice that the spectrum of a matrix is invariant under permutation similarity and transposition. Thus all zero-nonzero patterns that are equivalent to a spectrally arbitrary pattern are spectrally arbitrary.

An $n \times n$ zero-nonzero pattern $\mathcal{A}$ *allows nilpotency* if there exists a realization $N \in \mathcal{A}$ such that $p_N(t) = t^n$. We say that N is a *nilpotent realization* of $\mathcal{A}$. Let $\{f_1, f_2, ..., f_n\}$ denote the coefficient functions of the characteristic polynomial for a zero-nonzero pattern $\mathcal{A}$, so that

$$p_{\mathcal{A}}(t) = t^n + f_1 t^{n-1} + \ldots + f_{n-1}t + f_n.$$

Observe that each f_i is a function of the nonzero entries in $\mathcal{A}$. Given a subset $\{x_1, x_2, \ldots, x_n\} \subset \{a_{i,j} \in \mathcal{A} | a_{i,j} \neq 0\}$, a *Jacobi*, $\mathcal{J}$, is an $n \times n$ matrix with entries $\frac{\partial f_j}{\partial x_i}$. In this thesis a *Jacobian* of a zero-nonzero pattern $\mathcal{A}$ is the determinant of a Jacobi matrix formed from the coefficient functions of $p_{\mathcal{A}}(t)$. Notice that a zero-nonzero pattern may have many Jacobians depending on the choice of variables x_i.

2.2 The Nilpotent-Jacobian Condition

In [6] the authors develop the Nilpotent-Jacobian condition using the implicit function theorem. The condition works as follows: Let $\mathcal{A}$ be an irreducible zero-nonzero pattern. If possible, find a nilpotent realization $N \in \mathcal{A}$. Treating n of the nonzero elements of $\mathcal{A}$ as variables, find the Jacobian of the coefficient functions of $p_{\mathcal{A}}(t)$. If the Jacobian evaluated at N is nonzero, then $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary. If this can be done for at least one nilpotent realization $N \in \mathcal{A}$ and for at least one set of n nonzero entries in $\mathcal{A}$, we say $\mathcal{A}$ *satisfies the Nilpotent-Jacobian condition.*

Example 2.2.1 *Let* $\mathcal{A} = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}$*, then* $N = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ *is a nilpotent realization of* $\mathcal{A}$*. The coefficient functions of* $\mathcal{A}$ *are:*

$f_1 = -a_{1,1} - a_{3,3}$

$f_2 = -a_{2,1}a_{1,2} + a_{3,3}a_{1,1}$

$f_3 = -a_{1,2}a_{2,3}a_{3,1} + a_{3,3}a_{1,2}a_{2,1}$

Forming a Jacobi of $\mathcal{A}$ *with respect to the entries* $a_{1,1}, a_{2,1}, a_{3,1}$*, we have:*

$$J = \begin{bmatrix} -1 & a_{3,3} & 0 \\ 0 & -a_{1,2} & a_{3,3}a_{1,2} \\ 0 & 0 & -a_{1,2}a_{2,3} \end{bmatrix}.$$

The corresponding Jacobian is $det(J) = -a_{1,2}^2 a_{2,3}$. *This Jacobian evaluated at* N, *is* -1. *Thus* $\mathcal{A}$ *satisfies the Nilpotent-Jacobian condition. Hence* $\mathcal{A}$ *and all of its superpatterns are spectrally arbitrary.*

The implicit function theorem can be applied to complex analytic functions, thus the Nilpotent-Jacobian condition also applies to irreducible complex zero-nonzero patterns.

2.3 Graphs

The *directed graph* of an $n \times n$ zero-nonzero pattern $\mathcal{A} = \{a_{i,j} | 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}$, denoted $\mathcal{G}(\mathcal{A})$, is defined by $\mathcal{G}(\mathcal{A}) = (V, E(\mathcal{A}))$, where $V = \{1, 2, ..., n\}$ and $E(\mathcal{A}) = \{(i, j) | a_{i,j} \neq 0\}$.

Example 2.3.1 $\mathcal{G}(\mathcal{A})$ *for* $\mathcal{A} = \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & * \end{bmatrix}$ *is:*

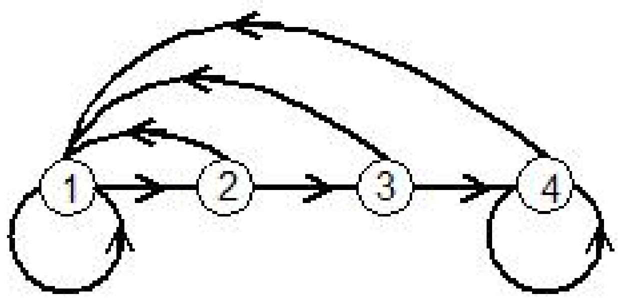

Figure 1: Directed Graph

A *path* in $\mathcal{G}(\mathcal{A})$ is a sequence of vertices $(v_1, v_2, ..., v_k)$ so that $(v_j, v_{j+1}) \in E(\mathcal{A})$ for $j = 1, 2, ..., k-1$. A *cycle* is a path such that $v_1 = v_k$. A *simple cycle* (resp. path), is a cycle (resp. path) where all but the first and last vertices are distinct. The number of edges in a simple cycle (resp. path) is the *length* of the cycle (resp. path). We define a *k-cycle*, as a simple cycle of length k. A pair of cycles, $c_1 = (v_1, v_2, \ldots, v_k)$ and $c_2 = (w_1, w_2, \ldots, w_j)$, are *disjoint* if $v_j \neq w_l$ for all $j = 1, 2, \ldots, k$ and $l = 1, 2, \ldots, j$. Given a collection of disjoint cycles γ, whose total length is k, we denote by τ_k the union of edges in γ. A *transversal* is a τ_n.

Example 2.3.2 *Let $\mathcal{G}(\mathcal{A})$ be the following:*

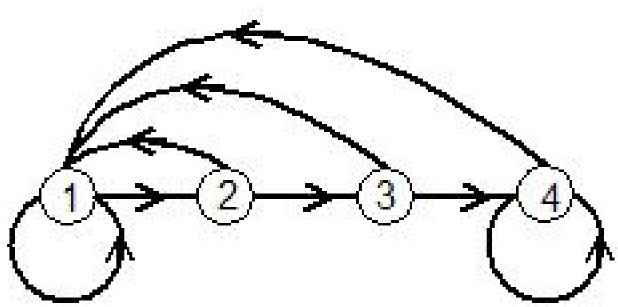

Figure 2: Directed Graph

Observe that $(1, 2)$ *is a simple path of length one,* $(1, 2, 1)$ *is a* 2*-cycle, and* $(1, 2, 3, 1)(4, 4)$ *is a transversal in* $\mathcal{G}(\mathcal{A})$.

Chapter 3

Complex Spectrally Arbitrary Patterns

3.1 Irreducible complex spectrally arbitrary patterns which do not satisfy the Nilpotent-Jacobian condition

Let $\mathcal{A}$ be an irreducible real zero-nonzero pattern which satisfies the Nilpotent-Jacobian condition. Let N be a nilpotent realization of $\mathcal{A}$ which corresponds to a nonzero Jacobian. If we consider the nonzero entries of $\mathcal{A}$ over $\mathbb{C}$, N will still remain a nilpotent realization of $\mathcal{A}$. Thus $\mathcal{A}$, viewed as an irreducible complex zero-nonzero pattern, will satisfy the Nilpotent-Jacobian condition. Hence, if an irreducible real zero-nonzero pattern satisfies the Nilpotent-Jacobian condition, then the corresponding irreducible complex zero-nonzero pattern also satisfies the Nilpotent-Jacobian condition.

In [4], a complete list of all irreducible 4×4 real spectrally arbitrary patterns is provided. In this thesis we amend this list to include additional zero-nonzero patterns that are spectrally arbitrary over $\mathbb{C}$, but not over $\mathbb{R}$. Of these patterns two are of particular interest and are discussed in the next lemma.

Lemma 3.1.1 *The following zero-nonzero patterns are irreducible complex spectrally arbitrary patterns whose Jacobians are zero at every nilpotent realization.*

$$\text{(i) } \mathcal{N}_4 = \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \end{bmatrix}$$

$$\text{(ii) } \mathcal{M}_4 = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix}$$

Proof:

(i) Observe that the coefficient functions for the characteristic polynomial of $\mathcal{N}_4$ are:

$f_1 = -a_{3,3} - a_{1,1}$

$f_2 = a_{1,1}a_{3,3} - a_{4,2}a_{2,4}$

$f_3 = a_{4,2}a_{2,4}a_{3,3} - a_{4,2}a_{2,3}a_{3,4} - a_{4,1}a_{1,2}a_{2,4} + a_{1,1}a_{4,2}a_{2,4}$

$f_4 = a_{4,1}a_{1,2}a_{2,4}a_{3,3} - a_{1,1}a_{4,2}a_{2,4}a_{3,3} - a_{4,1}a_{1,2}a_{2,3}a_{3,4} + a_{1,1}a_{4,2}a_{2,3}a_{3,4}$

By setting $f_1 = 0, f_2 = 0, f_3 = 0$, and $f_4 = 0$, we see that every nilpotent realization of $\mathcal{N}_4$ has the general form, $\begin{bmatrix} -a_{3,3} & \frac{-a_{4,2}a_{3,3}}{a_{4,1}} & 0 & 0 \\ 0 & 0 & \frac{-a_{3,3}^3}{a_{4,2}a_{3,4}} & \frac{-a_{3,3}^2}{a_{4,2}} \\ 0 & 0 & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & 0 & 0 \end{bmatrix}$.

We create an 8×4 matrix K with entries $\frac{\partial f_j}{\partial x_i}$, where the f_j are the coefficient functions described above and

$x_1 = a_{1,1}$; $x_2 = a_{1,2}$; $x_3 = a_{2,3}$; $x_4 = a_{2,4}$; $x_5 = a_{3,3}$; $x_6 = a_{3,4}$; $x_7 = a_{4,1}$; $x_8 = a_{4,2}$:

$$K = \begin{bmatrix} -1 & a_{3,3} & a_{4,2}a_{2,4} & a_{4,2}a_{2,3}a_{3,4} - a_{4,2}a_{2,4}a_{3,3} \\ 0 & 0 & -a_{4,1}a_{2,4} & a_{4,1}a_{2,4}a_{3,3} - a_{4,1}a_{2,3}a_{3,4} \\ 0 & 0 & -a_{4,2}a_{3,4} & -a_{4,1}a_{1,2}a_{3,4} + a_{1,1}a_{4,2}a_{3,4} \\ 0 & -a_{4,2} & a_{4,2}a_{3,3} - a_{4,1}a_{1,2} + a_{1,1}a_{4,2} & a_{4,1}a_{1,2}a_{3,3} - a_{1,1}a_{4,2}a_{3,3} \\ -1 & a_{1,1} & a_{4,2}a_{2,4} & a_{4,1}a_{1,2}a_{2,4} - a_{1,1}a_{4,2}a_{2,4} \\ 0 & 0 & -a_{4,2}a_{2,3} & -a_{4,1}a_{1,2}a_{2,3} + a_{1,1}a_{4,2}a_{2,3} \\ 0 & 0 & -a_{1,2}a_{2,4} & a_{1,2}a_{2,4}a_{3,3} - a_{1,2}a_{2,3}a_{3,4} \\ 0 & -a_{2,4} & a_{2,4}a_{3,3} - a_{2,3}a_{3,4} + a_{1,1}a_{2,4} & -a_{1,1}a_{2,4}a_{3,3} + a_{1,1}a_{2,3}a_{3,4} \end{bmatrix}$$

Observe that K has rank three at the general form of a nilpotent realization for $\mathcal{N}_4$. Thus treating any four nonzero entries of $\mathcal{N}_4$ as variables will result in a Jacobian that is zero at every nilpotent realization. Thus, $\mathcal{N}_4$ does not satisfy the Nilpotent-Jacobian condition. To show that $\mathcal{N}_4$ is a complex spectrally arbitrary pattern, we must prove it directly, which we do next.

Let c_1, c_2, c_3, c_4 be any elements in $\mathbb{C}$. Let $a_{1,2} = a_{2,4} = a_{3,4} = 1$, then the following values for $a_{1,1}, a_{4,2}, a_{4,1}, a_{2,3}$ will give a realization $N_4 \in \mathcal{N}_4$ with

$p_{N_4}(t) = t^4 + c_1t^3 + c_2t^2 + c_3t + c_4$:

$a_{1,1} = -a_{3,3} - c_1$;

$a_{4,2} = -a_{3,3}^2 - c_1a_{3,3} - c_2$;

$a_{4,1} = \left(c_2 + a_{3,3}^2 + c_1a_{3,3}\right)\left(a_{3,3} + c_1 + \frac{\sqrt{c_3^2 - 4a_{3,3}^2c_4 - 4c_1a_{3,3}c_4 - 4c_2c_4}}{2(a_{3,3}^2 + c_1a_{3,3} + c_2)}\right) - \frac{c_3}{2}$;

$a_{2,3} = a_{3,3} + \frac{c_3 + \sqrt{c_3^2 - 4a_{3,3}^2 c_4 - 4c_1 a_{3,3} c_4 - 4c_2 c_4}}{2(a_{3,3}^2 + c_1 a_{3,3} + c_2)}$;

Notice that we may choose $a_{3,3} \in \mathbb{R}$ large enough so that $a_{1,1}, a_{4,2}, a_{4,1}, a_{2,3}$ are all nonzero. Thus $\mathcal{N}_4$ is a complex spectrally arbitrary pattern.

(ii)Every nilpotent realization of $\mathcal{M}_4$ has the general form,

$$\begin{bmatrix} -a_{2,2} & \frac{-a_{2,2}a_{4,2}}{a_{4,1}} & a_{1,3} & 0 \\ \frac{a_{4,1}a_{2,2}}{a_{4,2}} & a_{2,2} & \frac{-a_{4,1}a_{1,3}}{a_{4,2}} & 0 \\ 0 & 0 & 0 & a_{3,4} \\ a_{4,1} & a_{4,2} & 0 & 0 \end{bmatrix}.$$

As with $\mathcal{N}_4$, this form is derived from setting $p_{\mathcal{M}_4}(t) = t^4$. We create a 9×4 matrix K with entries $\frac{\partial f_j}{\partial x_i}$, where

$f_1 = -a_{1,1} - a_{2,2}$

$f_2 = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$

$f_3 = -a_{4,1}a_{1,3}a_{3,4} - a_{4,2}a_{2,3}a_{3,4}$

$f_4 = -a_{4,1}a_{1,2}a_{2,3}a_{3,4} + a_{1,1}a_{4,2}a_{2,3}a_{3,4} + a_{4,1}a_{1,3}a_{3,4}a_{2,2} - a_{2,1}a_{4,2}a_{1,3}a_{3,4}$

and

$x_1 = a_{1,1}$; $x_2 = a_{1,2}$; $x_3 = a_{1,3}$; $x_4 = a_{2,1}$; $x_5 = a_{2,2}$; $x_6 = a_{2,3}$; $x_7 = a_{3,4}$; $x_8 = a_{4,1}$; $x_9 = a_{4,2}$,

Observe that K has rank three at the general form of a nilpotent realization for $\mathcal{M}_4$. Thus, as with $\mathcal{N}_4$, treating any four nonzero entries of $\mathcal{M}_4$ as variables will result in a Jacobian that is zero at every nilpotent realization. Thus, $\mathcal{M}_4$ does not satisfy the Nilpotent-Jacobian condition. To show that $\mathcal{M}_4$ is a complex spectrally arbitrary pattern, we must prove it directly, which we do next.

Let c_1, c_2, c_3, c_4 be any elements in $\mathbb{C}$. Let $a_{1,2} = a_{2,3} = a_{3,4} = a_{4,1} = 1$, then the following values for $a_{1,1}, a_{2,1}, a_{4,2}, a_{1,3}$ will give a realization $M_4 \in \mathcal{M}_4$ with $p_{M_4}(t) = t^4 + c_1 t^3 + c_2 t^2 + c_3 t + c_4$:

$a_{1,1} = -a_{2,2} - c_1$;

$a_{2,1} = -c_1 a_{2,2} - a_{2,2}^2 - c_2$;

$$a_{4,2} = \frac{-c_3}{2} + \frac{-2a_{2,2}-c_1}{2c_1 a_{2,2}+2a_{2,2}^2+2c_2} - \frac{\sqrt{a_{2,2}^4 c_3^2+2c_1 a_{2,2}^3 c_3^2+(-4c_4+2c_2 c_3^2+2c_1 c_3+c_1^2 c_3^2)a_{2,2}^2+(2c_1^2 c_3+2c_1 c_3^2 c_2-4c_1 c_4)a_{2,2}+c_1^2-4c_2 c_4+c_2^2 c_3^2-4c_2+2c_1 c_2 c_3}}{2c_1 a_{2,2}+2a_{2,2}^2+2c_2};$$

$$a_{1,3} = \frac{-c_3}{2} + \frac{2a_{2,2}+c_1}{2c_1 a_{2,2}+2a_{2,2}^2+2c_2} + \frac{\sqrt{a_{2,2}^4 c_3^2+2c_1 a_{2,2}^3 c_3^2+(-4c_4+2c_2 c_3^2+2c_1 c_3+c_1^2 c_3^2)a_{2,2}^2+(2c_1^2 c_3+2c_1 c_3^2 c_2-4c_1 c_4)a_{2,2}+c_1^2-4c_2 c_4+c_2^2 c_3^2-4c_2+2c_1 c_2 c_3}}{2c_1 a_{2,2}+2a_{2,2}^2+2c_2};$$

Notice that we may choose $a_{2,2} \in \mathbb{R}$ large enough so that $a_{1,1}, a_{2,1}, a_{4,2}, a_{1,3}$ are all nonzero. Thus $\mathcal{M}_4$ is a complex spectrally arbitrary pattern.$\square$

Lemma 3.1.1 immediately leads to the following observation for irreducible complex zero-nonzero patterns.

Observation 3.1.2 *Satisfying the Nilpotent-Jacobian condition is not necessary for an irreducible complex spectrally arbitrary zero-nonzero pattern.*

It should be noted that among all 4×4 complex zero-nonzero patterns, up to equivalence, $\mathcal{N}_4$ and $\mathcal{M}_4$ are the only irreducible complex spectrally arbitrary patterns which do not satisfy the Nilpotent-Jacobian condition. Another interesting consequence is that among all 4×4 complex spectrally arbitrary patterns, up to equivalence, $\mathcal{N}_4$ and $\mathcal{M}_4$ are the only complex spectrally arbitrary patterns which are not real spectrally arbitrary, yet have a real nilpotent realization.

Open Question 3.1.3 *Must every irreducible real spectrally arbitrary zero-nonzero pattern satisfy the Nilpotent-Jacobian condition?*

As shown in [[16], Theorem 6], this question is directly related to the $2n$-conjecture, stated next:

Conjecture 3.1.4 *Let $\mathcal{A}$ be an $n \times n$ irreducible spectrally arbitrary pattern. For each integer $n \geq 2$, $\mathcal{A}$ contains at least $2n$ nonzero entries.*

In [1] the authors prove that the minimum number of nonzero entries contained in an irreducible $n \times n$ real zero-nonzero pattern is at least $2n - 1$. The same proof is applicable to complex zero-nonzero patterns.

If the $2n$-conjecture is false, Theorem 6 in [16] implies that satisfying the Nilpotent-Jacobian is not a property of an irreducible spectrally arbitrary pattern with $2n - 1$ nonzero entries. All irreducible real spectrally arbitrary patterns classified thus far do indeed satisfy the Nilpotent-Jacobian condition and have no fewer than $2n$ nonzero entries. All complex spectrally arbitrary patterns classified thus far contain at least $2n$ nonzero entries.

The Nilpotent-Jacobian condition is a convenient method of proof for irreducible spectrally arbitrary patterns because it implies that any superpattern is also spectrally arbitrary. It is still unknown in general if every superpattern of a spectrally arbitrary pattern is also spectrally arbitrary. Therefore, a natural question for the reader to ask is if $\mathcal{N}_4$'s and $\mathcal{M}_4$'s superpatterns are also complex spectrally arbitrary. Indeed they are and proof is provided in Section 3.5.

3.2 Graphs of complex spectrally arbitrary patterns that do not contain a two-cycle

In [2], the authors prove that the graph of a real spectrally arbitrary must contain a two-cycle. The following displays that such a property is not required for a graph of a complex spectrally arbitrary pattern. We begin with the only (up to equivalency) 3×3 complex spectrally arbitrary zero-nonzero pattern whose graph does not contain a two-cycle.

Lemma 3.2.1 $\mathcal{D}_3 = \begin{bmatrix} * & * & 0 \\ 0 & * & * \\ * & 0 & * \end{bmatrix}$ *is a complex spectrally arbitrary pattern.*

<u>Proof:</u> The following is a nilpotent realization of $\mathcal{D}_3$, which corresponds to a nonzero Jacobian when the boxed entries are the x_i used to form the Jacobi.

$$D_3 = \begin{bmatrix} \boxed{\dfrac{-1-\sqrt{3}i}{2}} & 1 & 0 \\ 0 & \boxed{\dfrac{-1+\sqrt{3}i}{2}} & 1 \\ -1 & 0 & \boxed{1} \end{bmatrix}.$$

Thus this pattern satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{D}_3$, as well as all of its superpatterns, are complex spectrally arbitrary patterns.□

We continue with the classification of all 4×4 complex spectrally arbitrary patterns whose graphs do not contain a two-cycle. First we establish the following lemma.

Lemma 3.2.2 *Let $\mathcal{A}$ be an $n \times n$ complex spectrally arbitrary zero-nonzero pattern, with $n \geq 3$. If $\mathcal{G}(\mathcal{A})$ does not contain a two-cycle, then $\mathcal{G}(\mathcal{A})$ contains at least three loops.*

Proof: Let $\mathcal{A}$ be a complex spectrally arbitrary zero-nonzero pattern where $\mathcal{G}(\mathcal{A})$ does not contain a two-cycle. If $\mathcal{G}(\mathcal{A})$ does not contain a loop, then there exist no diagonal entries in $\mathcal{A}$. Hence, the coefficient function f_1 is always zero. If $\mathcal{G}(\mathcal{A})$ contains exactly one loop, then there exist one diagonal entry in $\mathcal{A}$. Hence, the coefficient function f_1 is always nonzero. Thus $\mathcal{G}(\mathcal{A})$ must have at least two loops. If $\mathcal{G}(\mathcal{A})$ has exactly two loops, then $\mathcal{A}$ has exactly two diagonal entries and the coefficient function f_2 is always nonzero. Hence $\mathcal{G}(\mathcal{A})$ must contain at least three loops.$\square$

Theorem 3.2.3 *Let $\mathcal{A}$ be an 4×4 irreducible complex spectrally arbitrary zero-nonzero pattern. If $\mathcal{G}(\mathcal{A})$ does not contain a two-cycle, then $\mathcal{A}$ is equivalent to a superpattern of one of the following patterns:*

$$\begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & * \\ 0 & * & * & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}.$$

Proof: We consider two main cases when classifying all irreducible 4×4 complex spectrally arbitrary patterns whose graphs contain at least three loops and no two-cycle;

(i) The graph contains a four-cycle.

(ii) The graph does not contain a four-cycle, but contains at least one three-cycle.

Suppose that $\mathcal{A}$ is an irreducible 4×4 complex zero-nonzero pattern whose graph contains exactly three loops and exactly one four-cycle, but no two-cycle. If no other edges are contained in $\mathcal{G}(\mathcal{A})$, then without loss of generality,

$$\mathcal{A} = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix}$$

Notice that $\mathcal{A}$ has seven nonzero entries and the determinant is $-a_{1,2}a_{2,3}a_{3,4}a_{4,1}$, which is always nonzero. Therefore $\mathcal{A}$ is not a complex spectrally arbitrary pattern. Up to equivalence, there are four superpatterns of $\mathcal{A}$ with exactly one additional nonzero entry and no two cycle:

$$\mathcal{A}_1 = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix}, \mathcal{A}_3 = \begin{bmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_4 = \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ * & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix}.$$

The following is a nilpotent realization of $\mathcal{A}_1$, which corresponds to a nonzero Jacobian when the boxed entries are the x_i used to form the Jacobi.

$$A_1 = \begin{bmatrix} \boxed{-i} & 1 & 0 & 0 \\ 0 & \boxed{i} & 1 & 0 \\ 0 & 0 & \boxed{-1} & 1 \\ \boxed{-1} & 0 & 0 & 1 \end{bmatrix}.$$

Thus this pattern satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{A}_1$, as well as all of its superpatterns, are complex spectrally arbitrary patterns.

The following is a nilpotent realization of $\mathcal{A}_2$, which corresponds to a nonzero Jacobian when the boxed entries are the x_i used to form the Jacobi.

$$A_2 = \begin{bmatrix} \boxed{\dfrac{-1-\sqrt{3}i}{2}} & 1 & 0 & 0 \\ 0 & \boxed{\dfrac{-1+\sqrt{3}i}{2}} & 1 & 1 \\ 0 & 0 & 1 & \boxed{1} \\ \boxed{-1} & 0 & 0 & 0 \end{bmatrix}.$$

Thus this pattern satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{A}_2$, as well as all of its superpatterns, are complex spectrally arbitrary patterns.

The following is a nilpotent realization of $\mathcal{A}_3$, which corresponds to a nonzero Jacobian when the boxed entries are the x_i used to form the Jacobi.

$$A_3 = \begin{bmatrix} \boxed{\dfrac{-1-\sqrt{3}i}{2}} & \boxed{\dfrac{-1+\sqrt{3}i}{2}} & \boxed{1} & 0 \\ 0 & \boxed{\dfrac{-1+\sqrt{3}i}{2}} & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus this pattern satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{A}_3$, as well as all of its superpatterns, are complex spectrally arbitrary patterns.

The determinant of $\mathcal{A}_4$ is $-a_{1,2}a_{2,3}a_{3,4}a_{4,1}$, which is always nonzero. Hence $\mathcal{A}_4$ is not a spectrally arbitrary pattern. Observe that any superpattern of $\mathcal{A}_4$ will either be equivalent to a superpattern of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, or its graph will contain a two-cycle.

This concludes our classification of all 4×4 complex spectrally arbitrary patterns whose graph does not contain has no two-cycle, one four-cycle, and at least three loops. We now consider 4×4 patterns whose graph does not contain a four-cycle or a two-cycle, but contains at least one three-cycle and at least three loops.

Suppose $\mathcal{A}$ is an irreducible 4×4 complex zero-nonzero pattern whose graph does not contain a four-cycle or two-cycle, but contains exactly one three-cycle and exactly three loops. There are two possible patterns for $\mathcal{A}$ (all others are equivalent):

$$\begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ whose determinant is zero.}$$

$$\begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{bmatrix}, \text{ whose determinant is } a_{4,4}a_{1,2}a_{2,3}a_{3,1}, \text{ which is nonzero.}$$

Since neither of these patterns are spectrally arbitrary, we consider their superpatterns. Observe, at least two new nonzero entries are required for any superpattern, of either of these patterns, to be irreducible. There are two superpatterns to consider with eight nonzero entries (all others are equivalent or contain a four-cycle or a two-cycle);

$$\mathcal{A}_5 = \begin{bmatrix} * & * & 0 & * \\ 0 & * & * & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}, \mathcal{A}_6 = \begin{bmatrix} * & * & 0 & * \\ 0 & * & * & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & 0 \end{bmatrix}$$

The following is a nilpotent realization of $\mathcal{A}_5$, which corresponds to a nonzero Jacobian when the boxed entries are the x_i used to form the Jacobi.

$$A_5 = \begin{bmatrix} \boxed{\dfrac{-1-\sqrt{3}i}{2}} & 1 & 0 & \boxed{\dfrac{-2}{-1+\sqrt{3}i}} \\ 0 & \boxed{\dfrac{-1+\sqrt{3}i}{2}} & 1 & 0 \\ \boxed{\dfrac{1-\sqrt{3}i}{-3+\sqrt{3}i}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus this pattern satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{A}_5$, as well as all of its superpatterns, are complex spectrally arbitrary patterns.

Observe the determinant of $\mathcal{A}_6$ is $a_{4,4}a_{1,2}a_{2,3}a_{3,1}$, which is always nonzero. Thus this pattern is not spectrally arbitrary. Observe that any superpattern of $\mathcal{A}_6$ will either be a superpattern of $\mathcal{A}_5$ or its graph will either contain a four-cycle or a two-cycle.$\square$

This concludes our classification of all 4×4 irreducible complex minimally spectrally arbitrary patterns whose graphs do not contain a two-cycle.

3.3 Reducible complex spectrally arbitrary patterns

Let $\mathcal{A}$ be an $n \times n$ reducible real zero-nonzero pattern, with an irreducible $k \times k$ block and an irreducible $m \times m$ block, where k and m are both odd. Notice each of these blocks contribute at least one linear factor to $p_{\mathcal{A}}(t)$. Yet there exists many monic polynomials of degree n, over $\mathbb{R}$, that contain at most one linear factor. Hence $\mathcal{A}$ is not a real spectrally arbitrary pattern. Therefore, a direct sum of real spectrally arbitrary patterns is not necessarily spectrally arbitrary. In contrast, we have the following observation in the complex case.

Observation 3.3.1 *If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are irreducible complex spectrally arbitrary patterns then $\mathcal{B} = \mathcal{A}_1 \bigoplus \mathcal{A}_2 \bigoplus \cdots \bigoplus \mathcal{A}_k$ is a complex spectrally arbitrary pattern.*

This is a direct result of the fact that every polynomial over $\mathbb{C}$ factors into linear components. It should be noted that Observation 3.3.1 and its converse are not true for real spectrally arbitrary patterns, as shown in [5]. The counterexample for the converse found in [5] does not provide a counterexample over $\mathbb{C}$. Indeed, both of the irreducible blocks used in this counterexample are complex spectrally arbitrary patterns.

Open Question 3.3.2 *If $\mathcal{B} = \mathcal{A}_1 \bigoplus \mathcal{A}_2 \bigoplus \cdots \bigoplus \mathcal{A}_k$ is a complex spectrally arbitrary pattern, are $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ all required to be complex spectrally arbitrary patterns?*

As shown in [4] Lemma 3.1, the only 4×4 reducible real spectrally arbitrary zero-nonzero patterns are equivalent to superpatterns of

$$\mathcal{F} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

This result holds for reducible complex spectrally arbitrary patterns as well.

3.4 A complete list of all 3×3 irreducible complex spectrally arbitrary patterns

The following theorem characterizes all irreducible 3×3 complex spectrally arbitrary patterns.

Theorem 3.4.1 *Let $\mathcal{A}$ be an irreducible 3×3 complex zero-nonzero pattern.*

(i) If $\mathcal{A}$ has five or fewer nonzero entries, then $\mathcal{A}$ is not spectrally arbitrary.

(ii) If $\mathcal{A}$ has six nonzero entries and is spectrally arbitrary, then $\mathcal{A}$ is minimally spectrally arbitrary and is equivalent to one of:

$$\mathcal{D}_1 = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{bmatrix}, \mathcal{D}_2 = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \mathcal{D}_3 = \begin{bmatrix} * & * & 0 \\ 0 & * & * \\ * & 0 & * \end{bmatrix}$$

(iii) If $\mathcal{A}$ has at least seven nonzero entries, at least two of which lie on the diagonal, then $\mathcal{A}$ is spectrally arbitrary, but not minimally spectrally arbitrary.

Theorem 3.4.1 is identical to Theorem 1.1 in [4], except for complex spectrally arbitrary patterns, $\mathcal{D}_3$ is added. Note that $\mathcal{D}_3$ appears in Lemma 3.2.1 in Section 3.2 of this thesis. The reader is encouraged to review [1, 4] for proof of the remaining parts.

It should be noted that there does not exist a 3×3 reducible complex spectrally arbitrary pattern. A nonzero 1×1 block requires a nonzero eigenvalue. A zero 1×1 block requires singularity. Hence all reducible 3×3 complex zero-nonzero patterns are not spectrally arbitrary.

3.5 A complete list of all 4×4 irreducible complex spectrally arbitrary patterns

The following theorem classifies all 4×4 irreducible complex spectrally arbitrary patterns. It should be noted that the majority of work needed to prove the next theorem was developed by Luisette Corpuz and Judith McDonald in [4].

Theorem 3.5.1 *Let $\mathcal{A}$ be an irreducible 4×4 complex zero-nonzero pattern.*

(i) If $\mathcal{A}$ has seven or fewer nonzero entries, then $\mathcal{A}$ is not spectrally arbitrary.

(ii) If $\mathcal{A}$ has eight nonzero entries and is spectrally arbitrary , then $\mathcal{A}$ is a minimal spectrally arbitrary pattern and is equivalent to one of the patterns presented in Appendix A.

(iii) If $\mathcal{A}$ has nine nonzero entries and is a minimal spectrally arbitrary, then it is

equivalent to one of the patterns presented in Appendix B. Patterns with nine nonzero entries that are superpatterns of the patterns in Appendix A are also spectrally arbitrary. If $\mathcal{A}$ is not spectrally arbitrary and has at least two nonzero entries along the diagonal then it is equivalent to one of the patterns listed in Appendix C.

(iv) If $\mathcal{A}$ has ten nonzero entries, with at least two on the diagonal, which is not spectrally arbitrary, then $\mathcal{A}$ is equivalent to one of the patterns listed in Appendix D. Otherwise, $\mathcal{A}$ is a spectrally arbitrary pattern, but is not minimal.

(v) If $\mathcal{A}$ has at least two nonzero diagonal entries and at least eleven nonzero entries, then $\mathcal{A}$ is a spectrally arbitrary pattern that is not minimal.

We only provide proof for patterns that are not spectrally arbitrary over $\mathbb{R}$. Patterns that are not listed, fail to be complex spectrally arbitrary patterns for the same reasons they fail to be real spectrally arbitrary patterns in [4].

Proof:

Part i) Follows from our analysis of directed graphs with no two-cycle in Section 3.2 and Lemma 3.4 in [4]. Notice the proof in [4] does not change if we allow the nonzero entries to be complex numbers.

Part ii) We have established the irreducible complex spectrally arbitrary patterns with eight nonzero entries, whose graph contains no two-cycle in Section 3.2. Reviewing the proof of Theorem 3.5 in [4] we note that up to equivalence, only two patterns are not real spectrally arbitrary, but are complex spectrally arbitrary. One of these patterns is $\mathcal{N}_4$, the other is

$$\mathcal{B}_1 = \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix}.$$

The following is a nilpotent realization of $\mathcal{B}_1$, which corresponds to a nonzero Jacobian when the boxed entries are the x_i used to form the Jacobi.

$$B_1 = \begin{bmatrix} \boxed{-1-i} & 1 & 0 & 0 \\ \boxed{-i} & 0 & 1 & 0 \\ 0 & 0 & \boxed{i} & 1 \\ \boxed{-1} & 0 & 0 & 1 \end{bmatrix}.$$

Thus this pattern satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{B}_1$, as well as all of its superpatterns, are complex spectrally arbitrary patterns.

This concludes our classification of all irreducible 4×4 complex spectrally arbitrary patterns for Part ii). All irreducible 4×4 complex minimally spectrally arbitrary patterns with eight nonzero entries may be found in Appendix A. Notice these patterns are indeed minimal, otherwise we would contradict Part i).

Part iii) Recall all superpatterns of irreducible patterns that satisfy the Nilpotent-Jacobian condition are spectrally arbitrary. Thus it has already been shown that any pattern with nine nonzero entries that is a superpattern of all but one ($\mathcal{N}_4$) of the complex spectrally arbitrary patterns with eight nonzero entries is a complex spectrally arbitrary pattern.

We will now consider the superpatterns of $\mathcal{N}_4$. As stated in [3] Proposition 2.4, any proper superpattern of $\mathcal{N}_4$ is a real spectrally arbitrary pattern. Hence every superpattern of $\mathcal{N}_4$ is complex spectrally arbitrary as well.

This concludes our classification of all irreducible 4×4 complex spectrally arbitrary patterns for Part iii). All irreducible 4×4 complex minimally spectrally arbitrary patterns with nine nonzero entries may be found in Appendix B. All irreducible 4×4 that are both minimal spectrally arbitrary patterns over $\mathbb{C}$ and $\mathbb{R}$ with nine nonzero entries may be found in Appendix B. It should be noted that patterns in [4] that were minimal over $\mathbb{R}$, which are not listed in Appendix B, were indeed superpatterns of one of the minimal complex spectrally arbitrary patterns with eight nonzero entries listed in Appendix A. All patterns that are irreducible with nine nonzero entries, at least two of which lie on the diagonal, that are not complex spectrally arbitrary are listed in Appendix C.

Part iv) It has been shown that any irreducible pattern with ten nonzero entries that is a superpattern of all but one ($\mathcal{M}_4$) of the complex spectrally arbitrary patterns with nine nonzero entries are complex spectrally arbitrary patterns. Indeed as shown in [3], Proposition 2.4, any proper superpattern of $\mathcal{M}_4$ is a spectrally arbitrary pattern.

This concludes our classification of all 4×4 irreducible complex spectrally arbitrary patterns for Part iv). Notice, none of these patterns can be minimal. All 4×4 irreducible patterns with ten nonzero entries, at least two of which lie on the diagonal, that are not complex spectrally arbitrary can be found in Appendix D.$\square$

Chapter 4

The minimum number of nonzero entries that guarantee a pattern is spectrally arbitrary

The goal of this chapter is to establish the following result.

Theorem 4.0.2 *If $\mathcal{A}$ is an irreducible $n \times n$ zero-nonzero pattern with $n > 6$ and at least $n^2 - 2n + 3$ nonzero entries at least two of which lie on the diagonal, then $\mathcal{A}$ is a spectrally arbitrary pattern.*

In this section we prove the following theorems, which establish Theorem 4.0.2.

Theorem 4.0.3 *Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern with $n > 6$ and at least $n^2 - 2n + 3$ nonzero entries, at least two of which lie on the diagonal. If $\mathcal{G}(\mathcal{A})$ does not contain an n-cycle, then $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary patterns.*

Theorem 4.0.4 *Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern with $n > 6$ and at least $n^2 - 2n + 3$ nonzero entries with two or more nonzero entries on the diagonal. If $\mathcal{G}(\mathcal{A})$ contains an n-cycle, then $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary patterns.*

This bound on the number of nonzero entries is strict, as the following example illustrates.

Example 4.0.5 *The following is an irreducible pattern with* $n^2 - 2n + 2$ *zeros which is not a spectrally arbitrary pattern:*

$$\begin{bmatrix} * & * & * & \cdots & * & * & * \\ * & * & * & \cdots & * & * & * \\ * & * & * & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * \\ * & 0 & 0 & \cdots & 0 & 0 & 0 \\ * & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

The determinant of this pattern is always zero. Thus this pattern is not spectrally arbitrary.

4.1 The case where $\mathcal{G}(\mathcal{A})$ does not contain an n-cycle

In this section we show if $\mathcal{A}$ is an irreducible zero-nonzero pattern (over $\mathbb{R}$ or $\mathbb{C}$) with exactly $n^2 - 2n + 3$ nonzero entries, at least two of which lie on the diagonal and $\mathcal{G}(\mathcal{A})$ does not contain a n-cycle, then $\mathcal{G}(\mathcal{A})$ must contain an $(n-1)$-cycle. We use this result to discover that all irreducible zero-nonzero patterns (over $\mathbb{R}$ or $\mathbb{C}$) with at least $n^2 - 2n + 3$ nonzero entries, at least two of which lie on the diagonal, are spectrally arbitrary patterns. We begin by establishing an upper bound for the length of the longest simple cycle contained in $\mathcal{G}(\mathcal{A})$.

Theorem 4.1.1 *Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern (over $\mathbb{R}$ or $\mathbb{C}$) with $n > 6$. Suppose $\mathcal{A}$ has at least $n^2 - 2n + 3$ nonzero entries, at least two of which lie on the diagonal. Then $\mathcal{G}(\mathcal{A})$ contains a simple cycle of length greater than $n - 2$.*

Proof: Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern with $n > 6$. Suppose $\mathcal{A}$ has $n^2 - 2n + 3$ nonzero entries, at least two of which lie on the diagonal. Let k be the length of the longest simple cycle contained in $\mathcal{G}(\mathcal{A})$. Without loss of generality, suppose $(1, 2, 3, \ldots, k, 1)$ is this simple cycle. Let $I = \{1, 2, \ldots, k\}$ and $J = \{k+1, k+2, \ldots, n\}$. Then for each $i \in I$ and each $j \in J$, if $(i, j) \in E(\mathcal{A})$ and $(j, (i+1) mod(k)) \in E(\mathcal{A})$, we can create a longer simple cycle. Hence, at least one these edges is missing from each such pair of indices. We have identified $k(n-k)$ zeros in our pattern. Since our pattern has at most $2n - 3$ zeros, $f(k) = k^2 - kn + 2n - 3$ must be nonnegative. Since this is a quadratic in k with $f(n-3) < 0$ and $f(3) < 0$, we see that either $k \leq 2$ or $k \geq n - 2$.

Notice there are $\frac{n!}{(n-3)!3}$ total 3-cycles in a complete graph on n vertices. Fix an edge (p, q) in a complete graph on n vertices. There are at most $n - 2$ vertices which can be paired with this edge to form a 3-cycle. Thus, removal of (p, q) from a complete graph removes at most $n - 2$ of the 3-cycles. Hence, in a complete graph the removal of at least $\frac{n!}{(n-3)!3(n-2)} = \frac{n(n-1)}{3}$ edges are required to ensure no 3-cycle remains. Since $\mathcal{A}$ has at most $2n - 3$ zero entries and $n > 6$, $\mathcal{G}(\mathcal{A})$ must contain at least one 3-cycle. Thus, the longest simple cycle contained in $\mathcal{G}(\mathcal{A})$ has length at least $n - 2$.

We finish the proof by showing the longest simple cycle in $\mathcal{G}(\mathcal{A})$ has length at least

$n-1$, via contradiction. Suppose the longest simple cycle in $\mathcal{G}(\mathcal{A})$ has length $n-2$. Without loss of generality, suppose $(1, 2, 3, \ldots, n-2, 1)$ is the longest simple cycle. Notice $I = \{1, 2, \ldots, n-2\}$ and $J = \{n-1, n\}$ and at least $2(n-2) = 2n-4$ zero entries have been identified in $\mathcal{A}$. Hence, there is only one remaining zero yet to be identified. Notice that the subgraph of $\mathcal{G}(\mathcal{A})$, induced by the vertices in I, is either the complete graph on $n-2$ vertices or the complete graph with the deletion of one edge. Similarly, the subgraph of $\mathcal{G}(\mathcal{A})$, induced by the vertices in J, is either the complete graph on 2 vertices or the complete graph with the deletion of one edge. Since $\mathcal{G}(\mathcal{A})$ is irreducible, we observe that there exist $i, l \in I$ and $\{j, m\} = \{n, n-1\}$ such that either (i, j, l) or (i, j, m, l) is a path contained in $\mathcal{G}(\mathcal{A})$. Clearly there is a simple path of length $n-3$, from l to i contained in the subgraph induced by I. This creates a simple cycle of length at least $n-1$, contradicting that the length of the longest simple cycle contained in $\mathcal{G}(\mathcal{A})$ is $n-2$. Hence, $\mathcal{G}(\mathcal{A})$ contains a simple cycle of length at least $n-1$.$\square$

We utilize this property to classify all irreducible zero-nonzero patterns, $\mathcal{A}$, with exactly $n^2 - 2n + 4$ nonzero entries, where $\mathcal{G}(\mathcal{A})$ does not contain an n-cycle. In the following Lemma, we establish two patterns for which all such $\mathcal{A}$ are equivalent.

Lemma 4.1.2 *Let $\mathcal{A}$ be an $n \times n$ irreducible zero-nonzero pattern (over $\mathbb{R}$ or $\mathbb{C}$) with $n^2 - 2n + 4$ nonzero entries, at least two of which lie on the diagonal. If $\mathcal{G}(\mathcal{A})$ does not contain an n-cycle, then $\mathcal{A}$ is equivalent to one of the following patterns:*

$$\mathcal{Y}_1 = \begin{bmatrix} * & * & \cdots & * & * & * \\ * & * & \cdots & * & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \cdots & * & * & 0 \\ * & * & \cdots & * & * & 0 \\ * & 0 & \cdots & 0 & 0 & * \end{bmatrix}, \mathcal{Y}_2 = \begin{bmatrix} * & * & * & \cdots & * & * & * \\ * & * & * & \cdots & * & * & 0 \\ * & 0 & * & \cdots & * & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & 0 & * & \cdots & * & * & 0 \\ * & 0 & * & \cdots & * & * & 0 \\ * & 0 & * & \cdots & * & * & * \end{bmatrix}$$

<u>Proof:</u> Let $\mathcal{A}$ be an $n \times n$ irreducible zero-nonzero pattern with $n^2 - 2n + 4$ nonzero entries, at least two of which lie on the diagonal. Suppose $\mathcal{G}(\mathcal{A})$ does not contain an n-cycle. By Lemma 4.1.1, $\mathcal{G}(\mathcal{A})$ contains at least one $(n-1)$-cycle. Without loss of generality let $(1, 2, 3, \ldots, n-2, n-1, 1)$ be an $(n-1)$-cycle contained in $\mathcal{G}(\mathcal{A})$. Let $I = \{1, 2, ..., n-1\}$. Notice that for $i \in I$ either $(i, n) \notin \mathcal{G}(\mathcal{A})$ or $(n, (i+1) mod(n-1)) \notin \mathcal{G}(\mathcal{A})$, identifying $n-1$ zeros in $\mathcal{A}$. $\mathcal{A}$ is irreducible, so there exists at least one $k \in I$ and at least one $m \in I$, such that $(k, n) \in E(\mathcal{A})$ and $(n, m) \in E(\mathcal{A})$. We will consider the two cases: $k = m$ and $k \neq m$.

<u>Case 1:</u> Suppose that the only choice for k and m forces $k = m$. Without loss of generality $k = m = 1$, $(i, n) \notin \mathcal{G}(\mathcal{A})$, and $(n, i) \notin \mathcal{G}(\mathcal{A})$, for $1 < i < n$. This identifies $2(n-2) = 2n - 4$ zeros in $\mathcal{A}$. Observe $\mathcal{A}$ is equivalent to $\mathcal{Y}_1$.

<u>Case 2:</u> Suppose k and m can be chosen so that $k \neq m$. Without loss of generality $k = 1$ and m can be chosen such that if $1 < i < m$, then $(n, i) \notin E(\mathcal{A})$ and $(i, n) \notin E(\mathcal{A})$. Notice this identifies an additional $m - 3$ zeros in $\mathcal{A}$.

If $m = 2$, then there exists an n-cycle: $(1, n, 2, 3, 4, \ldots, n-1, 1)$. Thus $m \neq 2$.

For each $p \in \{m, m+1, m+2, ..., n-1\}$ either $(p,2) \notin E(\mathcal{A})$ or $(m-1, (p+1)mod(n-1)) \notin E(\mathcal{A})$, otherwise $\mathcal{G}(\mathcal{A})$ contains the n-cycle: $(p, 2, 3, 4, \ldots, m-1, p+1, p+2, \ldots, n-1, 1, n, m, m+1, m+2, \ldots, p-1, p)$. This identifies an additional $n-1-m+1 = n-m$ zeros in $\mathcal{A}$. Thus $\mathcal{A}$ has $(n-1)+(m-3)+(n-m) = 2n-4$ zeros identified, leaving all other entries nonzero. If $m \neq 3$, then $\mathcal{G}(\mathcal{A})$ contains the n-cycle $(1, n, m, m-1, m-2, \ldots, 3, 2, m+1, m+2, \ldots, n-1, 1)$. Thus, $m = 3$.

We have established that edges not contained in $E(\mathcal{A})$ have the following forms: $(2, n)$, $(n, 2)$, exactly one of (i, n) or $(n, (i+1)mod(n-1))$ for $3 \leq i \leq n-1$, and exactly one of $(i, 2)$ or $(2, (i+1)mod(n-1))$, for $3 \leq i \leq n-1$. This identifies $2n-4$ zero entries in $\mathcal{A}$. Thus, all other edges are contained in $E(\mathcal{A})$. In particular, every vertex has a loop and the subgraph induced by vertices in $H = \{1, 3, 4, \cdots, n-1\}$ forms a complete graph on those $n-2$ vertices.

Let $p \in \{2, n\}$ and $q \in \{2, n\} \backslash \{p\}$. Suppose there exists $l, j \in \{4, 5, \ldots, n-1\}$ with $(l, p) \in E(\mathcal{A})$ and $(p, j) \in E(\mathcal{A})$. If $l = j$ then either there exists $r \neq j$ with $(p, r) \in E(\mathcal{A})$ or $s \neq i$ with $(s, p) \in E(\mathcal{A})$. Thus, we may choose vertices l and j such that $l \neq j$ and $\{(l, p), (p, j)\} \subset E(\mathcal{A})$. Without loss of generality, $l < j$. Notice there exists a simple path $(1, q, 3, 4, \ldots, l-1, l, p, j) \in \mathcal{G}(\mathcal{A})$. Since the subgraph induced by H forms a complete graph, clearly there exists a simple path from vertex j to vertex $n-1$, through the remaining vertices in $\mathcal{G}(\mathcal{A})$. Thus there exists the n-cycle $(1, q, 3, 4, \ldots, l-1, l, p, j, \ldots, n-1, 1) \in \mathcal{G}(\mathcal{A})$. Hence, either $(i, p) \in E(\mathcal{A})$ and $(p, (i+1)mod(n-1)) \notin E(\mathcal{A})$ for $i \in \{3, \ldots, n-1\}$ or $(i, p) \notin E(\mathcal{A})$ and $(p, (i+1)mod(n-1)) \in E(\mathcal{A})$ for $i \in \{3, \ldots, n-1\}$.

Let $p \in \{2, n\}$ and $q \in \{2, n\} \backslash \{p\}$. Suppose there exists $i \in \{3, \ldots, n-1\}$ such that $(q, i+1) \in E(\mathcal{A})$ and $(i, p) \in E(\mathcal{A})$. Then $(4, p) \in E(\mathcal{A})$ and $(q, 5) \in E(\mathcal{A})$. Again since the subgraph induced by H is a complete graph, there exists an n-cycle $(4, p, 3, 1, q, 5, 6, \ldots, n-1, 4) \in \mathcal{G}(\mathcal{A})$. Thus, $(2, i+1), (n, i+1) \in \mathcal{G}(\mathcal{A})$ for $i \in \{3, \ldots, n-1\}$, or $(i, 2), (i, n) \in \mathcal{G}(\mathcal{A})$ for $i \in \{3, \ldots, n-1\}$.

Hence $\mathcal{A}$ is equivalent to $\mathcal{Y}_2$. Thus, $\mathcal{A}$ is either equivalent to $\mathcal{Y}_1$ or $\mathcal{Y}_2$.$\square$

We proceed by proving that all irreducible $\mathcal{Y}_1^-$ and $\mathcal{Y}_2^-$ are spectrally arbitrary patterns. We begin with $\mathcal{Y}_1^-$. We frequently refer to the star pattern:

$$\mathcal{S} = \begin{bmatrix} * & * & * & \ldots & * \\ * & * & 0 & \ldots & 0 \\ * & 0 & * & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ * & 0 & 0 & \ldots & * \end{bmatrix}$$

Theorem 4.1.3 *Let $\mathcal{Y}_1$ be defined as in Lemma 4.1.2. If $\mathcal{Y}_1^-$ is irreducible, then $\mathcal{Y}_1^-$ and all of its superpatterns are spectrally arbitrary.*

Proof: Let $a_{p,q}$ be the nonzero entry in $\mathcal{Y}_1$ is zero in $\mathcal{Y}_1^-$. Let $\mathcal{S}$ be the star pattern with the loop missing from the center vertex. Notice that $\mathcal{Y}_1$ is a superpattern of a star pattern where the center of the star is vertex 1. In [12] they prove that a signing of $\mathcal{S}$ and all of its superpatterns are spectrally arbitrary patterns. This proves that the zero-nonzero pattern $\mathcal{S}$ and all of its superpatterns are spectrally arbitrary. If $a_{p,p} = 0$ with $p = 1$ or $a_{p,q} = 0$ with $p \neq 1$ and $q \neq 1$, then $\mathcal{Y}_1^-$ a superpattern of $\mathcal{S}$, where the center of the star is vertex 1. Also, if edge $(1, n) \notin E(\mathcal{Y}_1^-)$ or $(n, 1) \notin E(\mathcal{Y}_1^-)$, then $\mathcal{Y}_1^-$ is reducible. Thus we only consider the following cases for $a_{p,q} \in \mathcal{Y}_1^-$:

1. The entry $a_{p,q} = 0$ with $p = 1$, $q \neq 1$, and $q \neq n$. Without loss of generality $q \neq 2$.

2. The entry $a_{p,q} = 0$ with $p \neq 1$, $q = 1$, and $p \neq n$. Without loss of generality $q \neq 2$.

3. The entry $a_{p,q} = 0$ with $p = q$ and $p \neq 1$.

In each case the following is a directed graph corresponding to $\mathcal{B}$, a pattern similar to a subpattern of $\mathcal{Y}_1^-$:

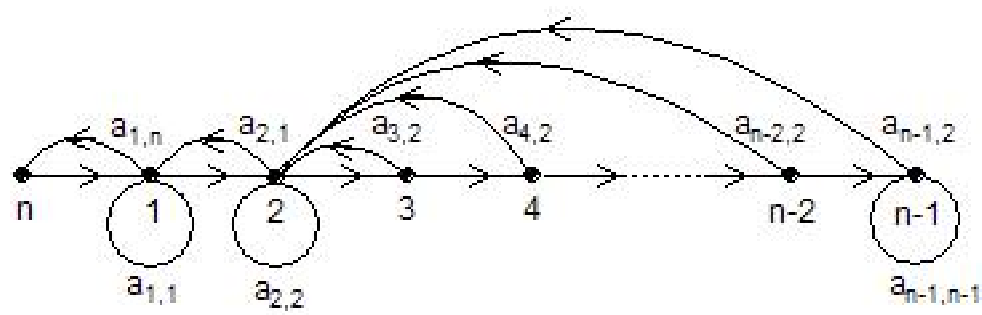

Figure 3: Subpattern of $\mathcal{Y}_1^-$

Setting $a_{k,k+1} = 1$ for $k = 1, 2, 3, 4, ..., n-2$ and $a_{n,1} = 1$, the coefficient functions for $\mathcal{B}$ are as follows:

$f_n = a_{1,n}(a_{n-1,2} - a_{n-1,n-1}a_{n-2,2});$

$f_{n-1} = a_{1,n}(a_{n-2,2} - a_{n-1,n-1}a_{n-3,2}) + a_{1,1}(a_{n-1,2} - a_{n-1,n-1}a_{n-2,2});$

$f_{n-2} = a_{1,n}(a_{n-3,2} - a_{n-1,n-1}a_{n-4,2}) + a_{1,1}(a_{n-2,2} - a_{n-1,n-1}a_{n-3,2})$

$+ a_{n-1,n-1}a_{n-2,2} - a_{n-1,2};$

$f_{n-k} = a_{1,n}(a_{n-k-1,2} - a_{n-1,n-1}a_{n-k-2,2}) + a_{1,1}(a_{n-k,2} - a_{n-1,n-1}a_{n-k-1,2})$

$+ a_{n-1,n-1}a_{n-k,2} - a_{n-k+1,2}$ for $n \geq k \geq 3;$

$f_4 = a_{1,n}(a_{3,2} - a_{n-1,n-1}a_{2,2}) + a_{1,1}(a_{4,2} - a_{n-1,n-1}a_{3,2})$

$+ a_{n-1,n-1}a_{4,2} - a_{5,2};$

$f_3 = a_{1,n}(a_{n-1,n-1} + a_{2,2}) + a_{n-1,n-1}(a_{3,2}$

$+ a_{2,1} - a_{2,2}a_{1,1}) - a_{4,2} + a_{3,2}a_{1,1};$

$f_2 = -a_{1,n} + a_{n-1,n-1}(a_{2,2} + a_{1,1}) - a_{3,2} + a_{2,2}a_{1,1} - a_{2,1};$

$f_1 = -a_{1,1} - a_{n-1,n-1} - a_{2,2};$

Solving for a_k in each of the equations $f_k = 0$ and applying back substitution, the following are nonzero values for a_k:

$a_{1,1} = -a_{n-1,n-1} - a_{2,2};$

$a_{1,n} = \dfrac{a_{n-1,n-1}(a_{n-1,n-1} + a_{2,2})^2}{a_{2,2}};$

$a_{2,1} = \dfrac{-(a_{n-1,n-1} + a_{2,2})^3}{a_{2,2}};$

$a_{k,2} = a_{n-1,n-1}^{k-2}a_{2,2}$ for $3 \leq k \leq n-1;$

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_{2,2}, a_{n-1,n-1}, a_{2,1}, a_{3,2}, a_{4,2}, a_{5,2}, ..., a_{n-1,2}$ evaluated at the nilpotent realization stated above and substituting

$a_{n-1,n-1} = 1$, $a_{2,2} = 1$, $a_{1,1} = 1$, $a_{1,n} = 4$, and $a_{2,1} = -8$, is $-(4)^{n-3}(-8)$. Thus $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{B}$, as well as all of its superpatterns, are complex spectrally arbitrary patterns.

Hence, all irreducible $\mathcal{Y}_1^-$ and their superpatterns are spectrally arbitrary patterns.□

We now proceed by proving that all irreducible $\mathcal{Y}_2^-$ in Lemma 4.1.2 are spectrally arbitrary patterns.

Theorem 4.1.4 *Let $\mathcal{Y}_2$ be defined as is Lemma 4.1.2. If $\mathcal{Y}_2^-$ is irreducible, then $\mathcal{Y}_2^-$ and all of its superpatterns are spectrally arbitrary.*

Proof: Let $a_{p,q}$ be the nonzero entry in $\mathcal{Y}_2$ which is zero in $\mathcal{Y}_2^-$. As in Theorem 4.1.3, $\mathcal{Y}_2$ is a superpattern of $\mathcal{S}$. If $a_{p,p} = 0$ with $p = 1$ or $a_{p,q} = 0$ with $p \neq 1$ and $q \neq 1$, then $\mathcal{Y}_2^-$ a superpattern of $\mathcal{S}$. If edge $(1, n) \notin E(\mathcal{Y}_2^-)$ or edge $(1, 2) \notin E(\mathcal{Y}_2^-)$, then $\mathcal{Y}_2^-$ is reducible. Since $\mathcal{Y}_2^-$ is symmetric in 2 and n, we can assume that there is a loop at n. Without loss of generality we assume that $(3, 1) \in E(\mathcal{A})$ and $(n-1, 1) \in E(\mathcal{A})$. We consider the following cases for $a_{p,q} \in \mathcal{Y}_2^-$:

1. The entry $a_{p,q} = 0$ with $p = 1$, $q \notin \{1, 2, n\}$.

2. The entry $a_{p,q} = 0$ with $p \neq 1$ and $q = 1$.

3. The entry $a_{p,q} = 0$ with $p = q$ and $p \neq 1$.

In each case the following is a directed graph corresponding to a pattern $\mathcal{B}$, similar to a subpattern of $\mathcal{Y}_2^-$:

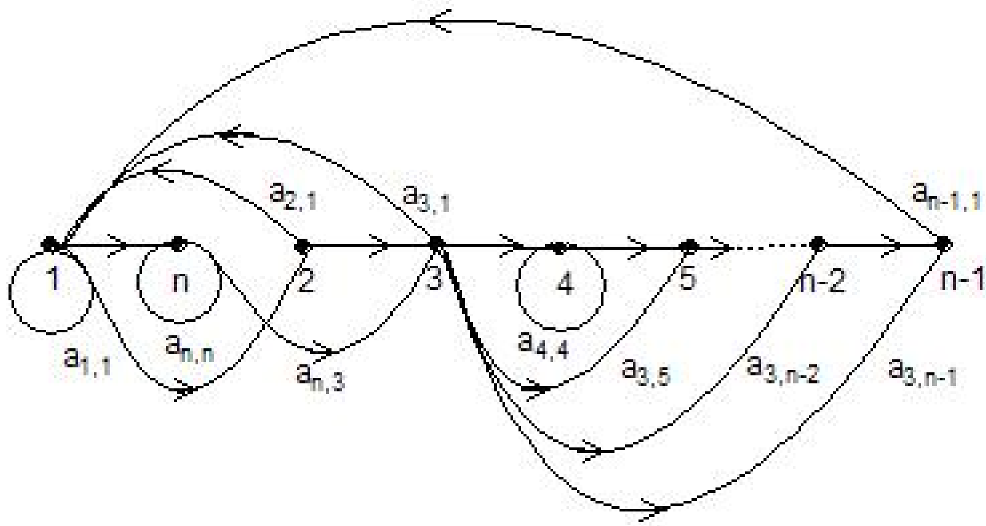

Figure 4: Subpattern of $\mathcal{Y}_2^-$

Setting $a_{k,k+1} = 1$ for $k = 1, 2, 3, 4, ..., n-2$ and $a_{1,n} = 1$, the coefficient functions for $\mathcal{B}$ are as follows:

$f_n = a_{n,n}a_{n-1,1}(1 - a_{3,5}a_{4,4})$;

$f_{n-1} = a_{n-1,1}((-1 + a_{3,5}a_{4,4})(1 + a_{n,3}) + a_{n,n}(a_{3,5} - a_{3,6}a_{4,4}))$;

$f_{n-2} = a_{n-1,1}((a_{n,3} + 1)(a_{3,6}a_{4,4} - a_{3,5}) + a_{n,n}(a_{3,6} - a_{3,7}a_{4,4}))$;

$f_{n-3} = a_{n-1,1}((a_{n,3} + 1)(a_{3,7}a_{4,4} - a_{3,6}) + a_{n,n}(a_{3,7} - a_{3,8}a_{4,4}))$;

$f_{n-k} = a_{n-1,1}((a_{n,3} + 1)(a_{3,k+4}a_{4,4} - a_{3,k+3}) + a_{n,n}(a_{3,k+4} - a_{3,k+5}a_{4,4}))$ for $n-4 \geq k \geq 5$;

$f_6 = a_{n-1,1}((a_{n,3} + 1)(a_{3,n-2}a_{4,4} - a_{3,n-3}) + a_{n,n}(a_{3,n-2} - a_{3,n-1}a_{4,4}))$;

$f_5 = a_{n-1,1}((a_{n,3} + 1)(a_{3,n-1}a_{4,4} - a_{3,n-2}) + a_{n,n}a_{3,n-1}) - a_{n,n}a_{4,4}a_{3,1}$;

$f_4 = (a_{n-1,1}a_{3,n-1} - a_{4,4}a_{3,1})(-a_{n,3} - 1) + a_{n,n}(a_{3,1} - a_{2,1}a_{4,4})$;

$f_3 = (-1 - a_{n,3})a_{3,1} + a_{n,n}(a_{2,1} - a_{4,4}a_{1,1}) + a_{4,4}a_{2,1}$;

$f_2 = a_{n,n}(a_{4,4} + a_{1,1}) + a_{4,4}a_{1,1} - a_{2,1}$;

$f_1 = -a_{1,1} - a_{n,n} - a_{4,4}$;

Solving for a_k in each of the equations $f_k = 0$ and applying back substitution, the following are the nonzero values for a_k:

$a_{3,k} = \dfrac{1}{a_{4,4}^{k-4}}$ for $5 \leq k \leq n-1$;

$a_{n-1,1} = -a_{4,4}^{n-3}(a_{n,n}a_{4,4} + a_{n,n}^2 + a_{4,4}^2)$;

$a_{3,1} = -a_{4,4}(a_{n,n}a_{4,4} + a_{n,n}^2 + a_{4,4}^2)$;

$a_{n,3} = \dfrac{a_{n,n}^3}{a_{4,4}(a_{n,n}a_{4,4} + a_{n,n}^2 + a_{4,4}^2)}$;

$a_{2,1} = -a_{n,n}a_{4,4} - a_{n,n}^2 - a_{4,4}^2$;

$a_{1,1} = -a_{n,n} - a_{4,4}$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_{1,1}, a_{4,4}, a_{2,1}, a_{3,1}, a_{3,5}, a_{3,6}, ..., a_{3,n-1}, a_{n-1,1}$ evaluated at the nilpotent realization stated above and substituting in $a_{n,n} = 1$, and $a_{4,4} = 1$, is $-2(-3)^{n-6}$. Thus $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{B}$, and all of its superpatterns, are complex spectrally arbitrary patterns.

Hence, all irreducible $\mathcal{Y}_2^-$ and their superpatterns are spectrally arbitrary.$\square$

With Theorems 4.1.3, 4.1.4, and Lemma 4.1.2 we may now state the following theorem.

Theorem 4.1.5 *Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern with $n > 6$ and at least $n^2 - 2n + 3$ nonzero entries, at least two of which lie on the diagonal. If $\mathcal{G}(\mathcal{A})$ does not contain an n-cycle, then $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary patterns.*

4.2 The case where $\mathcal{G}(\mathcal{A})$ contains an n-cycle

In this thesis, the $k \times k$ matrix $\mathcal{W}_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ a_2 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ a_{n-2} & 0 & 0 & \ddots & 1 & 0 \\ a_{n-1} & 0 & 0 & \cdots & 0 & 1 \\ a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$, where k indicates the size of $\mathcal{W}_k$. The weighted directed graph corresponding to $\mathcal{W}_k$ is illustrated below:

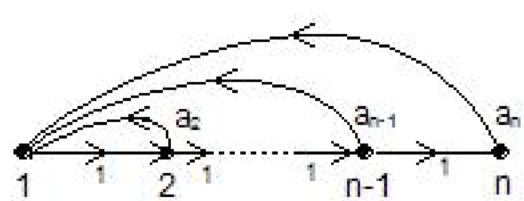

Figure 5: Fishbone Graph

We establish superpatterns of W_n, which will assist in proving Theorem 4.2.11. The proofs for the following Lemmas may be found in Appendix E.

Lemma 4.2.1 *Choose c and h such that $\frac{n+1}{2} \leq c \leq n$ and $c \leq h \leq n$. Let $\mathcal{B}$ be the superpattern of $\mathcal{W}_n$ created by making the following entries nonzero:*

- *The entry $b_{1,1} = a_1$.*
- *The entry $b_{n,c} = b_{n-c+1}$.*

- *The entry* $b_{h,h} = x$.

Then $\mathcal{B}$ *and all of its superpatterns are spectrally arbitrary.*

Lemma 4.2.2 *Choose* c, g *and* h *such that* $\frac{n+1}{2} \leq c \leq n-2$ *and* $1 \leq g < h \leq n$. *Let* $\mathcal{B}$ *be the superpattern of* $\mathcal{W}_n$ *created by making the following entries nonzero:*

- *Entry* $b_{n,c} = b_{n-c+1}$.
- *Entry* $b_{c+1,c} = b_2$.
- *Entry* $b_{c+2,c} = b_3$.
- *Entry* $b_{g,g} = a_1$.
- *Entry* $b_{h,h} = x$.

Then $\mathcal{B}$ *and all of its superpatterns are spectrally arbitrary.*

Lemma 4.2.3 *Choose* g, h *and* l *such that* $1 \leq g < h \leq n$ *and* $3 \leq l \leq n$. *Let* $\mathcal{B}$ *be the superpattern of* $\mathcal{W}_n$ *created by making the following entries nonzero:*

- *For all* $k \geq 3$, $b_{k,2} = b_{k-1}$.
- *Entry* $b_{g,g} = a_1$.
- *Entry* $b_{h,h} = x$.
- *Entry* $b_{n,l} = y$.

Then $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.

Lemma 4.2.4 *Choose g and h such that $1 \leq g < h \leq n$. Let $\mathcal{B}$ be the superpattern of $\mathcal{W}_n$ created by making the following entries nonzero:*

- *For all $k \geq 4$, $b_{k,3} = b_{k-2}$.*
- *Entry $b_{g,g} = a_1$.*
- *Entry $b_{h,h} = x$.*

Then $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.

Lemma 4.2.5 *Choose l such that $l \notin \{1, n-1\}$. Let $\mathcal{C}$ be a subpattern of $\mathcal{W}_n$ where entry $c_{2,1} = 0$. Let $\mathcal{B}$ be the superpattern of $\mathcal{C}$ created by making the following entries nonzero:*

- *Entry $b_{1,1} = a_1$.*
- *Entry $b_{n-1,n-1} = b_1$.*
- *Entry $b_{n,l} = x$.*
- *Entry $b_{1,n-1} = a_2$.*

Then $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.

Lemma 4.2.6 *Choose c, r, and l such that $\frac{n+1}{2} \leq c \leq n$, $n - c + 2 \leq r \leq n - 1$, and $l \notin \{1, c\}$. We require that if $c = n$, then $r \neq 2$. Let $\mathcal{C}$ be a subpattern of $\mathcal{W}_n$ where* *entry* $c_{r,1} = 0$. *Let* $\mathcal{B}$ *be the superpattern of* $\mathcal{C}$ *created by making the following entries nonzero:*

- *Entry* $b_{n,c} = b_{n-c+1}$.
- *Entry* $b_{1,1} = a_1$.
- *Entry* $b_{c,c} = b_1$.
- *Entry* $b_{n,l} = x$.

Then $\mathcal{B}$ *and all of its superpatterns are spectrally arbitrary.*

Lemma 4.2.7 *Choose* c, r, *and* l *such that* $\frac{n+1}{2} \leq c \leq n-1$, $2 \leq r \leq n-c+1$, *and* $l \notin \{1, c\}$. *Let* $\mathcal{C}$ *be a subpattern of* $\mathcal{W}_n$ *where entry* $c_{r,1} = 0$. *Let* $\mathcal{B}$ *be the superpattern of* $\mathcal{C}$ *created by making the following entries nonzero:*

- *Entry* $b_{n,c} = b_{n-c+1}$.
- *Entry* $b_{1,1} = a_1$.
- *Entry* $b_{c,c} = b_1$.
- *Entry* $b_{n,l} = y$.
- *Entry* $b_{c+r-1,c} = a_r$.

Then $\mathcal{B}$ *and all of its superpatterns are spectrally arbitrary.*

Lemma 4.2.8 *Choose* l *and* r *such that* $l \notin \{1, 3\}$ *and* $r > 4$. *Let* $\mathcal{C}$ *be a subpattern of*

$\mathcal{W}_n$ where entry $c_{2,1} = 0$. Let $\mathcal{B}$ be the superpattern of $\mathcal{C}$ created by making the following entries nonzero:

- *For all $k \notin \{1, 2, 3, 4, r\}$ $b_{k,2} = b_{k-1}$.*
- *Entry $b_{4,3} = a_2$.*
- *Entry $b_{1,1} = a_1$.*
- *Entry $b_{3,3} = b_1$.*
- *Entry $b_{n,l} = y$.*

Then $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.

We have classified all patterns needed to prove for Theorem 4.2.11. After the establishment of the following lemmas, we prove Theorem 4.2.11.

Lemma 4.2.9 *Let $\mathcal{A}$ be a superpattern of $\mathcal{W}_n$ with n^2-2n+3 nonzero entries and $n > 6$. If $\mathcal{A}$ has at least two strictly nonzero columns, then $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.*

Proof: Let $\mathcal{A}$ be a superpattern of $\mathcal{W}_n$ with n^2-2n+3 nonzero entries and $n > 6$. Suppose $\mathcal{A}$ has at least two strictly nonzero columns, c_1 and c_2. Create $\mathcal{G}(\mathcal{B})$ by relabeling the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i - c_1 + 1)mod(n)$. If this relabeling causes $(c_2 - c_1 + 1)mod(n) \leq \frac{n+1}{2}$, then relabel the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i - c_2 + 1)mod(n)$. In either case, $\mathcal{B}$ has the property that, column one in is strictly nonzero and there exists a column $c \geq \frac{n+1}{2}$ which is strictly nonzero. Thus $\mathcal{B}$ is a superpattern of the pattern in Lemma 4.2.1. Hence, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.□

Lemma 4.2.10 *Let $\mathcal{A}$ be a superpattern of $\mathcal{W}_n$ with n^2-2n+3 nonzero entries with $n>6$. Suppose that $\mathcal{A}$ has at least two nonzero entries along the diagonal. If row n of $\mathcal{A}$ contains at least $n-2$ zero entries, then $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.*

Proof: Let $\mathcal{A}$ be a superpattern of $\mathcal{W}_n$ with n^2-2n+3 nonzero entries with $n>6$. Suppose that $\mathcal{A}$ has at least two nonzero entries along the diagonal and row n contains at least $n-2$ zero entries. There are $2n-3$ zeros contained in $\mathcal{A}$, at least $n-2$ of which lie in row n. Thus there are at most $n-1$ zeros of $\mathcal{A}$ which are contained in rows 1 through $n-1$.

If there exist at least two strictly nonzero rows in $\mathcal{A}$, then $\mathcal{A}^T$ is equivalent to the pattern in Lemma 4.2.9.

Suppose there exists exactly one strictly nonzero row, d, in $\mathcal{A}$. Create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A}^T)$, such that vertex i is vertex $(i-d+1)mod(n)$. Observe, $\mathcal{B}$ has $n-3$ columns with exactly one zero entry, one column with two zero entries, one column with no zero entries, and one column with $n-2$ zero entries. Since $n>6$, at least $\frac{n+1}{2}-2$ columns with exactly one zero entry are greater than $\frac{n+1}{2}$.

If one of these columns with exactly one zero entry that are greater than $\frac{n+1}{2}$ has a zero diagonal, then $\mathcal{B}$ is equivalent to the transpose of the pattern in Lemma 4.2.2.

Suppose each of the columns with exactly one zero entry that are greater than $\frac{n+1}{2}$ has a nonzero diagonal. If the column with $n-2$ zero entries and the column with 2 zero entries are both larger than $\frac{n+1}{2}$, then all columns less than $\frac{n+1}{2}$ have exactly one zero entry.

If column three of $\mathcal{B}$ has a zero diagonal, then $\mathcal{B}$ is equivalent to the transpose of the pattern in Lemma 4.2.4.

Suppose column two of $\mathcal{B}$ has a zero diagonal. If there does not exists an $(n, l) \in E(\mathcal{B})$ with $l \notin \{1, 2\}$, then column n of $\mathcal{B}$ has $n-2$ zero entries. Thus, there exist a column in $\mathcal{A}$ with $n-2$ zero entries and a row with $n-2$ zero entries. We have identified at least $2n-5$ zero entries of $\mathcal{A}$. Therefore there are at least $n-3$ columns with at most one zero entry. Indeed the zero entry of these columns must be in row n. Since $n > 6$, we know there are at least two nonconsecutive columns (c_1, c_2) in $\mathcal{A}$ with a zero entry in row n. Create $\mathcal{G}(\mathcal{C})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$, such that vertex i is vertex $(i-c_k+1)mod(n)$. We choose $k \in \{1, 2\}$, such that vertex $c_j \geq \frac{n+1}{2}$ with $j \in \{1, 2\} \backslash \{k\}$. Since there exists an n-cycle, vertex $(n-c_k+1)mod(n) \neq n$ in $\mathcal{G}(\mathcal{C})$. Thus row n in $\mathcal{C}$ contains at most 2 zero entries. Column c_j in $\mathcal{C}$ has exactly one zero entry, which is not n. Thus, $(n, c_j), (c_j, c_j) \in E(\mathcal{C})$, there exists an $l \notin \{1, c_j\}$ with $(n, l) \in E(\mathcal{C})$ and for exactly one vertex $r \notin \{1, n\}$, $(r, 1) \notin E(\mathcal{C})$. Hence, $\mathcal{C}$ is equivalent to either the pattern in Lemma 4.2.6, or the pattern in Lemma 4.2.7.

Suppose both columns two and three have nonzero diagonals. Create $\mathcal{G}(\mathcal{C})$ by permuting the vertices of $\mathcal{B}$ such that vertex i is vertex $(i-2)mod(n)$. Then column $n-1$ in $\mathcal{C}$ is strictly nonzero and column one has exactly one zero entry off the diagonal. If there exists an $l \notin \{1, n-1\}$ such that $(n, l) \in E(\mathcal{C})$, then $\mathcal{C}$ is equivalent to the transpose of the pattern in either Lemma 4.2.6 or the pattern in Lemma 4.2.7.

If there does not exist an $l \notin \{1, n-1\}$ such that $(n, l) \in E(\mathcal{C})$, then $\mathcal{C}$ contains $n-2$ entries in row n. Thus, there exist a column in $\mathcal{A}$ with $n-2$ zero entries and a row with $n-2$ zero entries. We have identified at least $2n-5$ zero entries of $\mathcal{A}$. Therefore there are at least $n-3$ columns with at most one zero entry. Indeed the zero entry of these columns must be in row n. Since $n > 6$, we know there are at least two nonconsecutive columns (c_1, c_2) in $\mathcal{A}$ with a zero entry in row n. Create $\mathcal{G}(\mathcal{D})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$, such that vertex i is vertex $(i - c_k + 1)mod(n)$. We choose $k \in \{1, 2\}$, such that vertex $c_j \geq \frac{n+1}{2}$ with $j \in \{1, 2\} \backslash \{k\}$. Since there exists an n-cycle, vertex $(n - c_k + 1)mod(n) \neq n$ in $\mathcal{G}(\mathcal{D})$. Thus row n in $\mathcal{D}$ contains at most 2 zero entries. Column c_j in $\mathcal{D}$ has exactly one zero entry, which is not n. Thus, $(n, c_j), (c_j, c_j) \in E(\mathcal{D})$, there exists an $l \notin \{1, c_j\}$ with $(n, l) \in E(\mathcal{D})$ and for exactly one vertex $r \notin \{1, n\}$, $(r, 1) \notin E(\mathcal{D})$. Hence, $\mathcal{D}$ is equivalent to either the pattern in Lemma 4.2.6, or the pattern in Lemma 4.2.7.

If the column with $n-2$ zero entries and the column with 2 zero entries in $\mathcal{B}$ are not both larger than $\frac{n+1}{2}$, then there exist an $l \geq \frac{n+1}{2}$ such that $(n, l) \in E(\mathcal{B})$. We assumed each column in $\mathcal{B}$ with exactly one zero entry that is greater than $\frac{n+1}{2}$ has a nonzero diagonal. Thus, $\mathcal{B}$ is equivalent to the transpose of the pattern in Lemma 4.2.1.

Suppose there does not exists a strictly nonzero row of $\mathcal{A}$. Then there exists $n-1$ rows which contain exactly one zero entry.

Suppose there exists at least three rows whose one zero entry is on the diagonal.

Thus there exists two nonconsecutive rows (c_1, c_2) whose one zero entry is along the diagonal. Create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A}^T)$, such that vertex i is vertex $(i-c_k+1)mod(n)$. We choose $k \in \{1, 2\}$, such that vertex $c_j \geq \frac{n+1}{2}$ with $j \in \{1, 2\} \backslash \{k\}$. Thus $\mathcal{B}$ is equivalent to the transpose of the pattern in Lemma 4.2.2.

Suppose there exists at least $n-3$ rows whose one zero entry lies off the diagonal. If there exists an $l \geq \frac{n+1}{2}$ such that $(n, l) \in E(\mathcal{A}^T)$, then $\mathcal{A}^T$ is equivalent to the transpose of either the pattern in Lemma 4.2.6 or the pattern in Lemma 4.2.7.

If there does not exists an $l \geq \frac{n+1}{2}$ such that $(n, l) \in E(\mathcal{A}^T)$, then column n of $\mathcal{A}$ contains $\frac{n+1}{2}$ zero entries. Thus we have identified at least $n-3+\frac{n+1}{2}$ zero entries in $\mathcal{A}$. We assume $n > 6$, so there exists at least three columns which contain at most one zero entry and that zero entry must be in row n. Thus we can find two nonconsecutive columns (c_1, c_2) whose one zero entry is in row n. Create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$, such that vertex i is vertex $(i - c_k + 1)mod(n)$. We choose $k \in \{1, 2\}$, such that vertex $c_j \geq \frac{n+1}{2}$ with $j \in \{1, 2\} \backslash \{k\}$. Since there exists an n-cycle, vertex $(n - c_k + 1)mod(n) \neq n$ in $\mathcal{G}(\mathcal{B})$. Thus row n in $\mathcal{B}$ contains at most 2 zero entries. Column c_j in $\mathcal{B}$ has exactly one zero entry, which is not n. Thus, $(n, c_j), (c_j, c_j) \in E(\mathcal{B})$, there exists an $l \notin \{1, c_j\}$ with $(n, l) \in E(\mathcal{B})$ and for exactly one vertex $r \notin \{1, n\}$, $(r, 1) \notin E(\mathcal{B})$. Hence, $\mathcal{B}$ is equivalent to either the pattern in Lemma 4.2.6, or the pattern in Lemma 4.2.7.

Hence $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.$\square$

Theorem 4.2.11 *Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern with $n > 6$ and at least $n^2 - 2n + 3$ nonzero entries with two or more nonzero entries on the diagonal.*

If $\mathcal{G}(\mathcal{A})$ contains an n-cycle, then $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary patterns.

Proof: Let $\mathcal{A}$ be an irreducible $n \times n$ zero-nonzero pattern with $n > 6$ and at least $n^2 - 2n + 3$ nonzero entries with two or more nonzero entries on the diagonal. Let $\mathcal{G}(\mathcal{A})$ contain an n-cycle, without loss of generality, $(1, 2, \ldots, n, 1)$ is this n-cycle. Notice $\mathcal{A}$ is a superpattern of $\mathcal{W}_n$. We will consider the following cases to assist our proof:

1. There exists at least two strictly nonzero columns in $\mathcal{A}$.

2. There does not exists a strictly nonzero column in $\mathcal{A}$.

3. There exists exactly one strictly nonzero column in $\mathcal{A}$.

Case 1 If there exists at least two strictly nonzero columns in $\mathcal{A}$, then $\mathcal{A}$ is equivalent to the pattern in Lemma 4.2.1. Hence, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Case 2 Suppose there does not exists a strictly nonzero column in $\mathcal{A}$. If $n-2$ columns contain at least two zero entries and we require no column is strictly nonzero, then we have identified $2(n-2)+2 = 2n-2$ zero entries in $\mathcal{A}$. Thus, there exists at least three columns that contain exactly one zero entry. We consider the following subcases:

i) There exists at least two columns of $\mathcal{A}$ that contain exactly one zero entry and have a zero diagonal.

ii) There exists at least two columns of $\mathcal{A}$ that contain exactly one zero entry and have a nonzero diagonal.

Subcase i) Suppose there exists at least two columns of $\mathcal{A}$ that contain exactly one zero entry and have a zero diagonal. If these two columns are not consecutive, then $\mathcal{A}$ is equivalent to the pattern in Lemma 4.2.2. If these two columns, (c_1, c_1+1) are consecutive, then create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i - c_1 + 1) mod(n)$. If there exists a vertex $l \notin \{1, 2\}$ such that $(n, l) \in E(\mathcal{B})$, then $\mathcal{B}$ is equivalent to the pattern in Lemma 4.2.3. Since $\mathcal{A}$ is equivalent to $\mathcal{B}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If there does not exist a vertex $l \notin \{1, 2\}$ such that $(n, l) \in E(\mathcal{B})$, then row n has $n-2$ zero entries. Thus, by Lemma 4.2.10, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary

Subcase ii) Suppose there exists at least two columns of $\mathcal{A}$ that contain exactly one zero entry and have a nonzero diagonal. Without loss of generality, these two columns are column one and column c, where $c \geq \frac{n+1}{2}$. Choose r and s such that $(r, 1) \notin E(\mathcal{A})$ and $(s, c) \notin E(\mathcal{A})$.

Suppose $r > n - c + 1$.

Suppose $s \neq n$.

If $c \leq n - 1$, then $\mathcal{A}$ is equivalent to a superpattern of the pattern in Lemma 4.2.6. Hence $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose $s = n$. Since we assume column c has a nonzero diagonal, we need not consider $c = n$.

If $c \leq n-2$, we create a subpattern, $\mathcal{B}$ of $\mathcal{A}$, with the same form as the pattern in Lemma 4.2.6 except the entry $b_{n,c} = 0$ and the entry $b_{n-1,c} = b_{n-c}$. Employing a similar proof as that presented in Lemma 4.2.6, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Hence, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose $c = n-1$.

Suppose $r = 2$.

Suppose there exists an $l \notin \{1, n-1\}$, such that $(n,l) \in E(\mathcal{A})$. Since column $n-1$ has exactly one zero entry we know $(1, n-1) \in E(\mathcal{A})$. Thus $\mathcal{A}$ is equivalent to a superpattern of the pattern in Lemma 4.2.5.

Suppose there does not exist an $l \notin \{1, n-1\}$, such that $(n,l) \in E(\mathcal{A})$. Then row n contains $n-2$ zero entries. By Lemma 4.2.10, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $r \neq 2$, create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i+2)mod(n)$. If there exists an $l \notin \{1,3\}$, such that $(n,l) \in E(\mathcal{B})$, then $\mathcal{B}$ is a superpattern of the pattern in Lemma 4.2.8. Since $\mathcal{A}$ is equivalent to $\mathcal{B}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose there does not exist an $l \notin \{1,3\}$, such that $(n,l) \in E(\mathcal{B})$. Then row n contains $n-2$ zero entries. By Lemma 4.2.10, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Since $\mathcal{A}$ is equivalent to $\mathcal{B}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose $r \leq n-c+1$. Since we assume that column one has a nonzero diagonal, this implies that $c \leq n-1$.

Suppose $s \neq n$.

Suppose $s \neq c + r - 1$.

If there exists an $l \notin \{1, c\}$ such that $(n, l) \in E(\mathcal{A})$, then $\mathcal{A}$ is equivalent to a superpattern of the pattern in Lemma 4.2.7. Hence $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose there does not exist an $l \notin \{1, c\}$, such that $(n, l) \in E(\mathcal{A})$. Then row n contains $n - 2$ zero entries. By Lemma 4.2.10, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose $s = c + r - 1$.

If there exists an $l \notin \{1, c\}$ such that $(n, l) \in E(\mathcal{A})$, we create a subpattern, $\mathcal{B}$ of $\mathcal{A}$, with the same form as the pattern in Lemma 4.2.7 except the entry $b_{c+r-1,c} = 0$ and the entry $b_{c+r-1,c} = b_{r-1}$. Employing a similar proof as that presented in Lemma 4.2.7, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Hence, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose there does not exist an $l \notin \{1, c\}$, such that $(n, l) \in E(\mathcal{A})$. Then row n contains $n - 2$ zero entries. By Lemma 4.2.10, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose $s = n$.

Suppose $c \leq n - 2$

If there exists an $l \notin \{1, c\}$ such that $(n, l) \in E(\mathcal{A})$, we create a subpattern, $\mathcal{B}$ of $\mathcal{A}$, with the same form as the pattern in Lemma 4.2.7 except the entry $b_{n,c} = 0$ and the entry $b_{n-1,c} = b_{n-c}$. Employing a similar proof as that presented in Lemma 4.2.7, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Hence, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose there does not exist an $l \notin \{1, c\}$, such that $(n, l) \in E(\mathcal{A})$. Then row n contains $n - 2$ zero entries. By Lemma 4.2.10, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $c = n - 1$, then $r = 2$. Since we assume that column $n - 1$ has exactly one zero entry which we have identified as the last entry, $(1, n - 1) \in E(\mathcal{A})$. If there exists an $l \notin \{1, c\}$ such that $(n, l) \in E(\mathcal{A})$, then $\mathcal{A}$ is equivalent to a superpattern of the pattern in Lemma 4.2.5. Hence, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose there does not exist an $l \notin \{1, c\}$, such that $(n, l) \in E(\mathcal{A})$. Then row n contains $n - 2$ zero entries. By Lemma 4.2.10, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Case 3 There exists exactly one strictly nonzero column in $\mathcal{A}$.

Without loss of generality let column 1 be the strictly nonzero column. If each of the remaining columns contain two or more zero entries, then there would be at least $2(n - 1) = 2n - 2 > 2n - 3$ zero entries, a contradiction. Hence there exists a column c with at most one zero entry. Choose vertex j such that $(j, c) \notin E(\mathcal{A})$. We consider the following subcases:

i) Vertex $j = c$.

ii) Vertex $j \neq c$.

Subcase i) Suppose $j = c$.

Suppose $c = 2$. If there exists an $l \geq 3$ such that $(n, l) \in E(\mathcal{A})$, then $\mathcal{A}$ is equivalent to a superpattern of the pattern in Lemma 4.2.3. Thus, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose there does not exist an $l \geq 3$, such that $(n, l) \in E(\mathcal{A})$. Then row n contains $n - 2$ zero entries. By Lemma 4.2.10, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $c = 3$, then $\mathcal{A}$ is equivalent to a superpattern of the pattern in Lemma 4.2.4. Hence, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $4 \leq c < \frac{n+1}{2}$, create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i - c + 1) mod(n)$. Then $\mathcal{B}$ is equivalent to a superpattern of the pattern in Lemma 4.2.2. Thus, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Since $\mathcal{B}$ is equivalent to $\mathcal{A}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $\frac{n+1}{2} \leq c \leq n - 2$, $\mathcal{A}$ is equivalent to a superpattern of the pattern in Lemma 4.2.2. Thus, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $c = n - 1$, create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i + 2) mod(n)$. Then $\mathcal{B}$ is equivalent to a superpattern of the pattern in Lemma 4.2.4. Thus, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Since $\mathcal{B}$ is equivalent to $\mathcal{A}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $c = n$, create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i + 1) mod(n)$.

Suppose there exist an $l \geq 3$ such that $(n, l) \in E(\mathcal{B})$. Then $\mathcal{B}$ is equivalent to a superpattern of the pattern in Lemma 4.2.3. Thus, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Since $\mathcal{B}$ is equivalent to $\mathcal{A}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Suppose there does not exist an $l \geq 3$, such that $(n, l) \in E(\mathcal{B})$. Then row n contains $n-2$ zero entries. By Lemma 4.2.10, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary. Since $\mathcal{B}$ is equivalent to $\mathcal{A}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Subcase ii) Vertex $j \neq c$.

We considered $c = n$, in Subcase i), thus we will not consider $c = n$.

If $2 \leq c < \frac{n+1}{2}$, create $\mathcal{G}(\mathcal{B})$ by permuting the vertices of $\mathcal{G}(\mathcal{A})$ such that vertex i is vertex $(i - c + 1) mod(n)$. Label $1 - c - 1 = -c$ with $\hat{c}$ and observe that $\frac{n+1}{2} \leq \hat{c} \leq n - 1$ and column $\hat{c}$ is strictly nonzero. Thus, $\mathcal{B}$ is equivalent to a superpattern of a pattern in Case 2, subcase ii). Since $\mathcal{B}$ is equivalent to $\mathcal{A}$, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

If $\frac{n+1}{2} \leq c \leq n - 1$, then $\mathcal{A}$ is a superpattern of a pattern in Case 2, subcase ii). Hence, $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.

Hence $\mathcal{A}$ and all of its superpatterns are spectrally arbitrary.$\square$

Chapter 5

Properties of patterns that contain exactly one transversal

The bound of n^2-2n+3 nonzero entries needed to guarantee an irreducible pattern, $\mathcal{A}$, is spectrally arbitrary may be refined to $\frac{n(n+1)}{2}+1$ nonzero entries, if we require there exist a $\mathcal{A}^-$ which contains exactly one transversal. In this chapter we frequently refer to the pattern $\mathcal{L} = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$

Conjecture 5.0.12 *Let $\mathcal{A}$ be an $n \times n$ irreducible zero-nonzero pattern with exactly one transversal and $\frac{n(n+1)}{2}$ nonzero entries, at least two of which lie on the diagonal, then $\mathcal{A}^+$ and its superpatterns are spectrally arbitrary.*

The current proof of this result is quite lengthy, it is omitted from this thesis in hopes of refining it to a much shorter length in the near future. We include the necessary lemmas required to prove the theorem.

To begin, we show that if $\mathcal{A}$ has only one transversal and $\frac{n(n+1)}{2}$ nonzero entries, then

$\mathcal{A}$ is similar to $\mathcal{L}Q$, for some permutation matrix Q. There are several nice properties in the associated weighted digraph of $\mathcal{L}$. For example if $l_{i,j} = 0$, then $l_{j,i} \neq 0$. In $\mathcal{G}(\mathcal{L})$, if vertex i has access to vertex j, then $i \geq j$. Thus for $\mathcal{G}(\mathcal{L}Q)$, if vertex i has access to vertex j, then $i \geq s^{-1}(j)$, where s is the permutation which formed Q. We exploit this property frequently in the following work. First we establish which permutations create the appropriate number of nonzero entries on the diagonal.

Lemma 5.0.13 *$\mathcal{L}Q$ will have at least two nonzero entries along the diagonal whenever Q is not formed from the permutation $s = (123...(n-2)(n-1)n)$.*

Proof: Let $s \in S_n$. Recall that $(lq)_{i,j} = l_{i,s(j)}$. So, $\mathcal{L}Q$ will have a zero diagonal entry whenever $s(i) > i$. Suppose $\mathcal{L}Q$ has fewer than 2 nonzero diagonal entries. We claim that $s(n-k) = n-k+1$ for $k = 1, ..., n-1$. For $k = 1$, we must have that $s(n-1) > n-1$, which requires $s(n-1) = n$. Suppose that up to k $s(n-k) = n-k+1$. We must have that $s(n-(k+1)) = s(n-k-1) > n-k-1$. The only number which is larger than $n-k-1$, which has not been previously assigned is $n-k$, hence $s(n-k-1) = n-k$. Thus, for all $k = 1, ...n-1$, $s(n-k) = n-k+1$. Lastly, this requires $s(n) = 1$, producing our permutation s.

Conversely, if $s = (123...(n-2)(n-1)n)$ then $\mathcal{L}Q = \begin{bmatrix} 0 & 0 & \cdots & 0 & * \\ * & 0 & \cdots & 0 & * \\ & & & & \\ * & * & \cdots & 0 & * \\ * & * & \cdots & * & * \end{bmatrix}$ which has only one nonzero diagonal entry.$\square$

We have now established which permutations satisfy the two nonzero entries along the diagonal requirement of Theorem 5.0.12. Next we establish which permutations satisfy the irreducible requirement of Theorem 5.0.12.

Lemma 5.0.14 *$\mathcal{L}Q$ is reducible if and only if there exist $k < n$ such that $\{s(1), s(2), \ldots$ $\ldots, s(k)\} = \{1, 2, ..., k\}$, where s is the permutation which forms Q.*

Proof: Suppose that there exist $k < n$ such that $\{s(1), s(2), ..., s(k)\} = \{1, 2, ..., k\}$, where s is the permutation which forms Q. Then $\mathcal{G}(\mathcal{L}Q)$ has no edges from vertices $\{1, 2, ..., k\}$ to vertices $\{k+1, k+2, ..., n\}$. So $\mathcal{G}(\mathcal{L}Q)$ is reducible. Hence $\mathcal{L}Q$ is reducible as well.

Conversely, suppose $\mathcal{L}Q$ is reducible. Notice that $\{s(1), s(2), ..., s(k)\} = \{1, 2, ..., k\}$ if and only if $\{s^{-1}(1), s^{-1}(2), ..., s^{-1}(k)\} = \{1, 2, ..., k\}$. Since $\mathcal{L}Q$ is reducible, $\mathcal{G}(\mathcal{L}Q)$ is reducible. So we may partition $\mathcal{G}(\mathcal{L}Q)$ into two disjoint nonempty digraphs $\mathcal{I}$ and $\mathcal{J}$ such that there is no edge from $\mathcal{I}$ to $\mathcal{J}$. Every vertex in $\mathcal{G}(\mathcal{L}Q)$ has an edge to $s^{-1}(1)$, so $s^{-1}(1) \in \mathcal{I}$. Let p be the size of $\mathcal{I}$.

Suppose vertex $1 \in \mathcal{J}$. So $s^{-1}(1) \neq 1$. We claim that $s^{-1}(k) \in \mathcal{I}$ for all $k = 1, 2, ..., n$. Since $s^{-1}(1) \neq 1$ and all vertices except vertex 1 have access to $s^{-1}(2)$, $s^{-1}(1)$ has access to $s^{-1}(2)$. So, $s^{-1}(2) \in \mathcal{I}$. Suppose that $s^{-1}(k)$ has access to every vertex from $s^{-1}(1)$ up to $s^{-1}(k)$. So $s^{-1}(i) \in \mathcal{I}$ for $i = 1, 2, ..., k$. Now all vertices except 1,2,...,k have access to $s^{-1}(k+1)$. So there exist a vertex in $\mathcal{I}$ with access to $s^{-1}(k+1)$. Thus $s^{-1}(k+1) \in \mathcal{I}$. Hence, $s^{-1}(k) \in \mathcal{I}$ for $k = 1, 2, ..., n$. A contradiction, since $\mathcal{J}$ and $\mathcal{I}$ are disjoint and nonempty.

Thus $1 \in \mathcal{I}$. Suppose that there is no $k < n$ such that $\{s^{-1}(1), s^{-1}(2), ..., s^{-1}(k)\} = \{1, 2, ..., k\}$. Recall $s^{-1}(1) \in \mathcal{I}$. We claim that $s^{-1}(i) \in \mathcal{I}$ for all $i = 1, 2, ..., n$. If $\{s^{-1}(1)\} \neq \{1\}$, then $s^{-1}(1)$ has access to vertex $s^{-1}(2)$. This implies that $s^{-1}(2) \in \mathcal{I}$. Suppose that $s^{-1}(i) \in \mathcal{I}$ for all $i = 1, 2, ..., k$. Now if $\{s^{-1}(1), s^{-1}(2), ..., s^{-1}(k)\} \neq \{1, 2, ..., k\}$ then at least one of $\{s^{-1}(1), s^{-1}(2), ..., s^{-1}(k)\}$ must have an edge to $s^{-1}(k+1)$. Hence, $s^{-1}(k+1) \in \mathcal{I}$. Thus, $\{s^{-1}(1), s^{-1}(2), ..., s^{-1}(i)\} \in \mathcal{I}$ for all $i = 1, 2, ..., n$. Contradiction, $\mathcal{J}$ and $\mathcal{I}$ are disjoint and nonempty. Thus, there must exist a $k < n$ such that $\{s^{-1}(1), s^{-1}(2), ..., s^{-1}(k)\} = \{1, 2, ..., k\}$.$\square$

We have established which permutations do not allow $\mathcal{L}Q$ to satisfy the two nonzero diagonal entries and irreducible requirements of Theorem 5.0.12. We use these results to prove that all such $\mathcal{A}$ which satisfy the conditions in Theorem 5.0.12 will be similar to $\mathcal{L}Q$ were Q is one of our allowed permutations.

Lemma 5.0.15 *For every $n \times n$ zero-nonzero pattern $\mathcal{A}$ with exactly one transversal, $\frac{n(n+1)}{2}$ nonzero entries, two of which lie on the diagonal, there exists a permutation matrix Q such that $\mathcal{A} \sim \mathcal{L}Q$.*

Proof: If $\mathcal{A}$ has a transversal, then there exist a permutation matrix P that is a subpattern of $\mathcal{A}$. Letting $Q = P^{-1}$, $\mathcal{A}Q$ places this transversal in $\mathcal{A}$, onto the diagonal of $\mathcal{A}Q$. Suppose $\mathcal{A}$ is a 2×2. So $\mathcal{A}Q$ is either upper or lower triangular. If $\mathcal{A}Q$ is upper triangular let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Notice that $P\mathcal{A}QP^T = \mathcal{L}$, or $\mathcal{A}Q \sim \mathcal{L}$.

Suppose $\mathcal{A}Q \sim \mathcal{L}$ for any $\mathcal{A}$ that is $k \times k$ with $k = 1, 2, ..., n$. Let $\mathcal{A}$ be $(n+1) \times (n+1)$. Notice that $(aq)_{i,j} = 0$ if and only if $(aq)_{j,i} \neq 0$. Otherwise, $\mathcal{A}Q$ would have an extra transversal or too few nonzero entries. Partition $\mathcal{A}$ as follows; $\mathcal{A}Q = \begin{bmatrix} b & c \\ d & \mathcal{A}_1 \end{bmatrix}$, where $b \in \mathbf{R}$, $c^T \in \mathbf{R}^{n-1}$, $d \in \mathbf{R}^{n-1}$, and $\mathcal{A}_1 \in \mathbf{R}^{(n-1)\times(n-1)}$. $\mathcal{A}_1$ satisfies the inductive hypothesis, so there exist a $P_1 \in \mathbf{R}^{(n-1)\times(n-1)}$ such that if $P = \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix}$, then $P\mathcal{A}QP^{-1} \sim \begin{bmatrix} b & c \\ d & \mathcal{L} \end{bmatrix}$. Suppose that there exist $i < j$ such that $c_i = 0$ and $c_j \neq 0$. This forces $d_i \neq 0$ and $d_j = 0$. This gives the 3-cycle, $(1(j+1)(i+1))$, which we may pair with $n-3$ loops to get a transversal. Contradiction, of the number of transversals contained in $\mathcal{A}$.

Thus, if $c_j \neq 0$, then $c_k \neq 0$ for all $1 \leq k \leq j-1$. Choose largest such j. Let $D_{j+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$ be in $\mathbf{R}^{(j+1)\times(j+1)}$ and $D = \begin{bmatrix} D_{j+1} & 0 \\ 0 & I_{n-j-1} \end{bmatrix}$. Then $DP\mathcal{A}Q(DP)^T = \mathcal{L}$, or $\mathcal{A}Q \sim \mathcal{L}$.$\square$

It is well known that a necessary condition for a pattern to be spectrally arbitrary is that the corresponding graph contain at least two transversals. We show that $\mathcal{G}(\mathcal{A}^+)$ will indeed have at least two transversals.

Lemma 5.0.16 *If $\mathcal{A}$ is an irreducible $n \times n$ zero-nonzero pattern with $\frac{n(n+1)}{2}$ nonzero* *entries at least two along the diagonal and $\mathcal{A} \sim \mathcal{L}Q$ for some permutation matrix Q, then adding a nonzero entry to $\mathcal{L}Q$, adds a transversal to $\mathcal{G}(\mathcal{A})$.*

Proof: Let $(lq)_{i,j} = l_{i,s(j)}$ be the entry which is now nonzero. Since this entry must lie in the upper triangular portion of $\mathcal{L}$, we see that $s(j) > i$. So we have a new transversal in $\mathcal{L}^+$, namely $\{(is(j)), (s(j)i), (1,1), (2,2), .., (i-1, i-1)(i+1, i+1), ..., (s(j)-1, s(j)-1), (s(j)+1, s(j)+1), ..., (n-1, n-1), (n, n)\}$. Thus, we have at least two transversals in $\mathcal{L}^+$. So, there are at least two permutation matrices P_1 and P_2 that are subpatterns of $\mathcal{L}^+$. Then P_1Q and P_2Q are subpatterns of $(\mathcal{L}Q)^+$. If $\mathcal{A} = S\mathcal{L}QS^T$, then SP_1QS^T and SP_2QS^T are subpatterns of $\mathcal{A}^+$. Hence $\mathcal{G}(\mathcal{A}^+)$ has at least two transversals. Thus, adding a nonzero entry to $\mathcal{L}Q$ adds at least one transversal to any $\mathcal{G}(\mathcal{A})$ where, $\mathcal{A} \sim \mathcal{L}Q$.$\square$

Chapter 6

Conclusions and Future Work

Further study of complex zero-nonzero patterns is encouraged. As shown in this thesis, complex patterns have many interesting properties not found in real patterns. Differences in studying spectrally arbitrary patterns between the real and complex cases appear to arise from the fact that all polynomials factor linearly over $\mathbb{C}$. The study of complex zero-nonzero patterns will lead to a more complete understanding of real zero-nonzero patterns. Of particular interest is to study the Nilpotent-Jacobian method and Implicit Function Theorem in more depth. Perhaps revealing whether or not it is a necessary condition for real spectrally arbitrary patterns.

As stated in the introduction, Theorem 4.2.3 holds for complex zero-nonzero patterns. Future study will be investigation into the relationship between the bound on the number of nonzero entries and the cycle structure of the underlining coefficient functions of $p_{\mathcal{A}}(t)$. Lastly establishing how this bound is effected by viewing the nonzero entries of $\mathcal{A}$ over $\mathbb{C}$.

Lastly refinement of the proof of Theorem 5.0.5 is a future goal. Future work will be investigation into how the number of transversals contained in a pattern effects the bound on the number of nonzero entries needed to guarantee that $\mathcal{A}$ is a spectrally arbitrary pattern.

APPENDIX A

Irreducible minimally spectrally arbitrary patterns with eight nonzero entries

Complex minimally spectrally arbitrary patterns that are not real spectrally arbitrary patterns

$$\begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * & 0 & * \\ 0 & * & * & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & * \end{bmatrix}$$

Patterns that are both complex minimally spectrally arbitrary and real minimally spectrally arbitrary

$$\begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & * \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & * & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & * & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & * & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & * & * \end{bmatrix} \begin{bmatrix} * & * & 0 & * \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ 0 & * & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & * \\ 0 & 0 & * & * \\ 0 & * & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ 0 & * & * & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ 0 & * & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ 0 & * & 0 & * \end{bmatrix} \begin{bmatrix} * & 0 & 0 & * \\ * & 0 & * & * \\ 0 & 0 & * & * \\ 0 & * & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & * \end{bmatrix}$$

APPENDIX B

Irreducible minimally spectrally arbitrary patterns with nine nonzero entries

Complex minimally spectrally arbitrary patterns that are not real spectrally arbitrary patterns

$$\begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix}$$

Patterns that are both complex minimally spectrally arbitrary and real minimally spectrally arbitrary

$$\begin{bmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & * \\ 0 & * & 0 & * \\ * & 0 & 0 & * \end{bmatrix} \begin{bmatrix} 0 & * & * & 0 \\ * & 0 & * & 0 \\ * & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix} \begin{bmatrix} 0 & * & * & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & * & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & * \\ * & 0 & 0 & * \\ * & * & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & * & 0 & 0 \\ 0 & * & * & * \\ * & 0 & 0 & * \\ * & * & 0 & * \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 0 & * \\ 0 & * & * & 0 \end{bmatrix} \begin{bmatrix} 0 & * & 0 & 0 \\ * & * & * & 0 \\ 0 & 0 & * & * \\ 0 & * & * & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & * \\ 0 & 0 & * & * \\ 0 & * & * & * \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & 0 \end{bmatrix} \begin{bmatrix} 0 & * & * & * \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & 0 & 0 & * \end{bmatrix}$$

APPENDIX C

Irreducible patterns with nine nonzero entries that are

NOT complex (real) spectrally arbitrary.

In addition to this list, any patterns with nine nonzero entries that have a pattern from Appendix D as a superpattern, are NOT complex spectrally arbitrary.

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & * & * \\ * & 0 & 0 & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & 0 \\ * & 0 & * & * \\ 0 & 0 & 0 & * \\ * & 0 & 0 & * \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & * & * & * \\ 0 & * & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & * & * \\ 0 & * & * & * \\ 0 & * & 0 & * \end{bmatrix} \begin{bmatrix} * & 0 & 0 & * \\ * & * & * & * \\ 0 & 0 & 0 & * \\ 0 & * & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & * & 0 \end{bmatrix} \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & * & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & * & 0 \end{bmatrix}$$

APPENDIX D

Irreducible patterns with ten nonzero entries that are

NOT complex (real) spectrally arbitrary.

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & * \\ * & * & * & * \\ * & 0 & 0 & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ * & 0 & * & 0 \\ * & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & * & 0 \\ * & 0 & * & 0 \\ * & * & * & * \\ * & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & * & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \\ 0 & * & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & * & 0 & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & 0 & * \end{bmatrix} \begin{bmatrix} 0 & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & * & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & * & 0 & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & 0 \end{bmatrix} \begin{bmatrix} 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & * & * & 0 \end{bmatrix}$$

APPENDIX E

Proofs for Lemmas 4.2.1-4.2.8

Lemma 4.2.1

Proof Choose c and h such that $\frac{n+1}{2} \leq c \leq n$ and $c \leq h \leq n$. Let $\mathcal{B}$ be the superpattern of $\mathcal{W}_n$ created by making the following entries nonzero:

- The entry $b_{1,1} = a_1$.
- The entry $b_{n,c} = b_{n-c+1}$.
- The entry $b_{h,h} = x$.

The following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - x$;

$f_k = -a_k + q_k$ for $2 \leq k \leq n-c$;

$f_{n-c+1} = -a_{n-c+1} - b_{n-c+1} + q_{n-c+1}$;

$f_k = -a_k + b_{n-c+1}a_{k-n+c-1} + q_k$ for $n-c+2 \leq k \leq n$;

For each of the coefficient functions f_k, q_k is the sum of the following terms when defined:

xa_{k-1} for $2 \leq k \leq h$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ evaluated at any nilpotent realization is $(-1)^n$. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, then this pattern and all of its superpatterns are spectrally arbitrary.

Solving for each a_k in $f_k = 0$ and applying back substitution, $a_k = -x^k$ for $1 \leq k \leq h$. For $h+1 \leq k \leq n$, a_k is a polynomial in x of degree $k-n+c-1$. Thus, there exists a nilpotent realization of $\mathcal{B}$. Therefore $\mathcal{B}$ satisfies the Nilpotent-Jacobian method. Hence, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.$\square$

Lemma 4.2.2

Proof Choose c, g and h such that $\frac{n+1}{2} \leq c \leq n-2$ and $1 \leq g < h \leq n$. Let $\mathcal{B}$ be the superpattern of $\mathcal{W}_n$ created by making the following entries nonzero:

- Entry $b_{n,c} = b_{n-c+1}$.
- Entry $b_{c+1,c} = b_2$.
- Entry $b_{c+2,c} = b_3$.
- Entry $b_{g,g} = a_1$.
- Entry $b_{h,h} = x$.

The following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - x;$

$f_2 = -a_2 - b_2 + q_2;$

$f_3 = -a_3 - b_3 + q_3;$

$f_4 = -a_4 + b_2a_2 + q_4;$

$f_k = -a_k + b_2a_{k-2} + b_3a_{k-3} + q_k$ for $5 \leq k \leq n-c;$

$f_{n-c+1} = -a_{n-c+1} - b_{n-c+1} + b_2a_{n-c-1} + b_3a_{n-c-2} + q_{n-c+1};$

$f_{n-c+2} = -a_{n-c+2} + b_2a_{n-c} + b_3a_{n-c-1} + q_{n-c};$

$f_k = -a_k + b_2a_{k-2} + b_3a_{k-3} + b_{n-c+1}a_{k-n+c-1} + q_k$ for $5 \leq k \leq c+1;$

$f_{c+2} = -a_{c+2} + b_3a_{c-1} + b_{n-c+1}a_{2c-n+1} + q_{c+2};$

$f_k = -a_k + b_{n-c+1}a_{k-n+c-1} + q_k$ for $c+3 \leq k \leq n$;

For each of the coefficient functions f_k, q_k is the sum of the following terms when defined:

xy for $k = 2$;

xa_{k-1} for $3 \leq k \leq g$ if $g \geq 3$;

$-xya_{k-2}$ for $4 \leq k \leq g+1$ if $g \geq 3$;

xb_2 for $k = 3$ if $g < c$;

xb_3 for $k = 4$ if $g < c$;

xb_{n-c+1} for $k = n-c+2$ if $g < c$;

$-xb_2a_{k-3}$ for $5 \leq k \leq g+2$ if $3 \leq g < c$;

$-xb_3a_{k-4}$ for $6 \leq k \leq g+3$ if $3 \leq g < c$;

$-xb_{n-c+1}a_{k-n+c-1}$ for $n-c+3 \leq k \leq g+n-c$ if $3 \leq g < c$;

$-xyb_2$ for $k = 4$ if $h < c$;

$-xyb_3$ for $k = 5$ if $h < c$;

$-xyb_{n-c+1}$ for $k = n-c+3$ if $h < c$;

xyb_2a_{k-3} for $5 \leq k \leq g+2$ if $3 \leq g < h < c$;

xyb_3a_{k-4} for $6 \leq k \leq g+3$ if $3 \leq g < h < c$;

$xyb_{n-c+1}a_{k-n+c-1}$ for $n-c+3 \leq k \leq g+n-c$ if $3 \leq g < h < c$;

ya_{k-1} for $3 \leq k \leq h$ if $h \geq 3$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ evaluated at any nilpotent realization is $(-1)^n$. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, then this pattern and all of its superpatterns are spectrally arbitrary.

Solving for each a_k in $f_k = 0$ and applying back substitution we achieve the following expressions for a_k:

Suppose $c \leq h$.

For $n-c+1 \leq k \leq n$, let $m = \lfloor \frac{c}{n-c+1} \rfloor$. Then a_k is a polynomial in b_{n-c+1} of degree i for $k = i(n-c+1)+p$, $1 \leq i \leq m$, and $0 \leq p \leq n-c$ and degree $m+1$ otherwise.

If $g \in \{1,2\}$, then $a_k = -x^k$ for $1 \leq k \leq n-c$.

If $g \notin \{1,2\}$, the following are nonzero expressions for a_k:

$a_1 = -x$;

$a_2 = -x^2$;

$a_3 = -b_3$;

For $4 \leq k \leq n-c$ a_k is a polynomial in x of degree k if k is even and of degree $k-3$ if k is odd.

Suppose $c > h$.

If $g \in \{1,2\}$, then $a_k = -x^k$ for $1 \leq k \leq min\{h, n-c\}$.

If $h = min\{h, n-c\}$, then for $h < k \leq n-c$, a_k is a polynomial in x of degree $h - (i)mod(2)$ where $k = h+i$.

For $n-c+1 \leq k \leq n$, let $m = \lfloor \frac{c}{n-c+1} \rfloor$. Then a_k is a polynomial in b_{n-c+1} of degree i for $k = i(n-c+1)+p$, $1 \leq i \leq m$, and $0 \leq p \leq n-c$ and degree $m+1$ otherwise.

If $n-c = min\{h, n-c\}$, for $n-c+1 \leq k \leq n$, let $m = \lfloor \frac{c}{n-c+1} \rfloor$. Then a_k is a polynomial in b_{n-c+1} of degree i for $k = i(n-c+1)+p$, $1 \leq i \leq m$, and $0 \leq p \leq n-c$ and degree $m+1$ otherwise.

If $g \notin \{1,2\}$, the following are nonzero expressions for a_k:

$a_1 = -x$;

$a_2 = -x^2$;

$a_3 = -b_3$;

For $4 \leq k \leq min\{h, n-c\}$, $a_k = -x^k$ for k even and $a_k = -x^{k-3}$ for k odd.

If $h = min\{h, n-c\}$, for $1 \leq i \leq n-c-h$ and $k = h+i$, a_k is a polynomial in x of degree $h-(i)mod(2)$ where if k is even and of degree $h-(i)mod(2)-3$ if k is odd.

For $n-c+1 \leq k \leq n$, let $m = \lfloor \frac{c}{n-c+1} \rfloor$. Then a_k is a polynomial in b_{n-c+1} of degree i for $k = i(n-c+1)+p$, $1 \leq i \leq m$, and $0 \leq p \leq n-c$ and degree $m+1$ otherwise.

If $n-c = min\{h, n-c\}$, for $n-c+1 \leq k \leq n$, let $m = \lfloor \frac{c}{n-c+1} \rfloor$. Then a_k is a polynomial in b_{n-c+1} of degree i for $k = i(n-c+1)+p$, $1 \leq i \leq m$, and $0 \leq p \leq n-c$ and degree $m+1$ otherwise.

Therefore, we can choose x and b_{n-c+1} such that a nilpotent realization of $\mathcal{B}$ exists. Thus $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{B}$, and all of its superpatterns, are spectrally arbitrary.$\square$

Lemma 4.2.3

<u>Proof</u> Choose g, h and l such that $1 \leq g < h \leq n$ and $3 \leq l \leq n$. Let $\mathcal{B}$ be the superpattern of $\mathcal{W}_n$ created by making the following entries nonzero:

- For all $k \geq 3$, $b_{k,2} = b_{k-1}$.

- Entry $b_{g,g} = a_1$.

- Entry $b_{h,h} = x$.

- Entry $b_{n,l} = y$.

Then the following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - x$;

$f_k = -a_k - b_k + q_k$ for $2 \leq k \leq n-1$;

$f_n = -a_n + q_n$;

For each of the coefficient functions f_k, q_k is the sum of the following terms when defined:

$a_1 x$ for $k = 2$;

$a_1 a_{k-1}$ for $3 \leq k \leq g$ and $g \geq 3$;

$x a_{k-1}$ for $3 \leq k \leq h$ and $h \geq 3$;

$-x a_1 a_{k-2}$ for $4 \leq k \leq g+1$ and $g \geq 3$;

$a_1 b_{k-1}$ for $3 \leq k \leq g-1$ and $g \geq 4$;

$x b_{k-1}$ for $3 \leq k \leq h-1$ and $h \geq 4$;

$-x a_1 b_{k-2}$ for $4 \leq k \leq g$ and $g \geq 4$;

$-y$ for $k = n-l+1$;

$a_1 y$ for $k = n-l+2$ and $g < l$;

xy for $k = n-l+2$ and $h < l$;

$-x a_1 y$ for $k = n-l+3$ and $h < l$;

$y a_{k-n+l-1}$ for $n-l+3 \leq k \leq n$;

$y b_{k-n+l}$ for $n-l+2 \leq k \leq n-2$ if $l \geq 4$;

$-a_1 y a_{k-n+l-2}$ for $n-l+4 \leq k \leq n-l+g+1$ if $l \geq g+1$;

$-x y a_{k-n+l-2}$ for $n-l+4 \leq k \leq n-l+h+1$ if $l \geq h+1$;

$-a_1 y b_{k-n+l-2}$ for $n-l+4 \leq k \leq n-l+g$ if $g \geq 4$ and $l \geq g+1$;

$-xyb_{k-n+l-2}$ for $n-l+4 \leq k \leq n-l+h$ if $h \geq 4$ and $l \geq h+1$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ evaluated at any nilpotent realization is $(-1)^n$. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, then this pattern and all of its superpatterns are spectrally arbitrary.

Solving for each a_k in $f_k = 0$ and applying back substitution the following are nonzero expressions for a_k:

$a_1 = -x$;

For $2 \leq k \leq n-1$ each a_k is a linear function in b_k and a_n is a linear function in y. Therefore, we can choose y and b_k for $2 \leq k \leq n-1$ such that a nilpotent realization of $\mathcal{B}$ exists. Thus, $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence, $\mathcal{B}$, and all of its superpatterns are spectrally arbitrary.□

Lemma 4.2.4

Proof Choose g and h such that $1 \leq g < h \leq n$. Let $\mathcal{B}$ be the superpattern of $\mathcal{W}_n$ created by making the following entries nonzero:

- For all $k \geq 4$, $b_{k,3} = b_{k-2}$.
- Entry $b_{g,g} = a_1$.
- Entry $b_{h,h} = x$.

The following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - x$;

$f_2 = -a_2 - b_2 + q_2$;

$f_k = -a_k - b_k + a_2 b_{k-2} + q_k$ for $3 \leq k \leq n-2$;

$f_{n-1} = -a_{n-1} + a_2b_{n-3} + q_{n-1}$;

$f_n = -a_n + a_2b_{n-2} + q_n$;

For each of the coefficient functions f_k, q_k is the sum of the following terms when defined:

a_1a_{k-1} for $3 \leq k \leq g$ if $g \geq 3$;

xa_{k-1} for $3 \leq k \leq h$ if $h \geq 3$;

$-a_1xa_{k-2}$ for $4 \leq k \leq g+1$ if $g \geq 3$;

a_1b_{k-1} for $3 \leq k \leq g-2$ if $g \geq 5$;

xb_{k-1} for $3 \leq k \leq h-2$ if $h \geq 5$;

$-a_1xb_{k-2}$ for $4 \leq k \leq g-1$ if $g \geq 5$;

$-a_1a_2b_{k-3}$ for $5 \leq k \leq g$, if $g \geq 5$;

$-xa_2b_{k-3}$ for $5 \leq k \leq h$ if $h \geq 5$;

$a_1xa_2b_{k-4}$ for $6 \leq k \leq g+1$ if $g \geq 5$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, a_2, \ldots, a_{n-1}, a_n$ evaluated at any nilpotent realization is $(-1)^n$. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, this pattern and all of its superpatterns are spectrally arbitrary.

Solving for a_k in each of the equations $f_k = 0$ and applying back substitution, the following are nonzero expressions for a_k:

$a_1 = -x$;

Observe that for $2 \leq k \leq n$, a_k is a linear function in b_k. Thus, we can choose x, and b_k for $2 \leq k \leq n-2$, such that a nilpotent realization of $\mathcal{B}$ exists. Thus $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{B}$, and all of its superpatterns, are spectrally arbitrary.□

Lemma 4.2.5

Proof Choose l such that $l \notin \{1, n-1\}$. Let $\mathcal{C}$ be a subpattern of $\mathcal{W}_n$ where entry $c_{2,1} = 0$. Let $\mathcal{B}$ be the superpattern of $\mathcal{C}$ created by making the following entries nonzero:

- Entry $b_{1,1} = a_1$.
- Entry $b_{n-1,n-1} = b_1$.
- Entry $b_{n,l} = x$.
- Entry $b_{1,n-1} = a_2$.

The following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - b_1;$

$f_2 = a_1 b_1 + q_2;$

$f_3 = -a_3 + q_3;$

$f_k = -a_k + b_1 a_{k-1} + q_k$ for $4 \leq k \leq n-1;$

$f_n = -a_n + q_n;$

For each of the coefficient functions f_k, q_k is the sum of the following terms when defined:

$-a_2 a_{n-1}$ for $k = 2;$

$-a_2 a_n$ for $k = 3;$

$-a_2 a_{k+l-4}$ for $4 \leq k \leq n-l+2;$

$-x$ for $k = n-l+1;$

$xa_{k-n+l-1}$ for $n-l+2 \leq k \leq n$

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, a_2, a_3 \ldots, a_{n-1}, a_n$ evaluated at any nilpotent realization is $(-1)^n a_n - g$, where g is a function of x, a_2 and b_1. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, this pattern and all of its superpatterns are spectrally arbitrary.

Solving for a_k in each of the equations $f_k = 0$ and applying back substitution, the following are nonzero expressions for a_k:

$a_1 = -b_1$;

$y = \frac{-b_1^2}{a_{n-1}}$;

$a_3 = \frac{a_n b_1^2}{a_{n-1}}$;

Notice a_k is a polynomial in b_1 of degree $k-1$ for $4 \leq k \leq n-1$. Thus, we can choose b_1 and x such that a nilpotent realization of $\mathcal{B}$ exists. Thus $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{B}$, and all of its superpatterns, are spectrally arbitrary.□

Lemma 4.2.6

Proof Choose c, r, and l such that $\frac{n+1}{2} \leq c \leq n$, $n-c+2 \leq r \leq n-1$, and $l \notin \{1, c\}$. We require that $c = n$ if and only if $r \neq 2$. Let $\mathcal{C}$ be a subpattern of $\mathcal{W}_n$ where entry $c_{r,1} = 0$. Let $\mathcal{B}$ be the superpattern of $\mathcal{C}$ created by making the following entries nonzero:

- Entry $b_{n,c} = b_{n-c+1}$.
- Entry $b_{1,1} = a_1$.
- Entry $b_{c,c} = b_1$.
- Entry $b_{n,l} = x$.

The following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - b_1$;

$f_k = -a_k + b_1 a_{k-1} + q_k$ for $2 \leq k \leq n-c$;

$f_{n-c+1} = -a_{n-c+1} + b_1 a_{n-c} - b_{n-c+1} + q_{n-c+1}$;

$f_k = -a_k + b_1 a_{k-1} + b_{n-c+1} a_{k-n+c-1} + q_k$ for $n-c+2 \leq k \leq r-1$;

$f_r = b_1 a_{r-1} + b_{n-c+1} a_{r-n+c-1} + q_r$;

$f_k = -a_k + b_1 a_{k-1} + b_{n-c+1} a_{k-n+c-1} + q_k$ for $r \leq k \leq c-1$;

$f_k = -a_k + b_{n-c+1} a_{k-n+c-1} + q_k$ for $c \leq k \leq n$;

In each f_k, the q_k are the sum of the following terms where defined:

$-x$ for $k = n-l+1$;

$x a_{k-n+l-1}$ for $n-l+2 \leq k \leq n$;

$x b_1$ for $k = n-l+2$ if $l > c$;

$-x b_1 a_{k-n+l-2}$ for $n-l+3 \leq k \leq n-l+c+1$ if $l > c$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, b_1, a_2, a_3, \ldots, a_{r-1}$, $a_{r+1}, a_{r+2}, \ldots, a_n$ evaluated at any nilpotent realization is $(-1)^n 2 b_1^{c-2} + g$, where g is a function of x, b_{n-c+1} and b_1. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, this pattern and all of its superpatterns are spectrally arbitrary.

Solving for a_k in each of the equations $f_k = 0$ and applying back substitution, for $1 \leq k \leq r-1$, a_k is a polynomial in b_1 of degree k. We choose to solve for b_{n-c+1} in $f_r = 0$, setting $b_{n-c+1} = b_1^{n-c+1}$. For $r+1 \leq k \leq n$, a_k is a polynomial in b_1 of degree k. Thus, we can choose x, b_1 such that a nilpotent realization of $\mathcal{B}$ exists. Thus $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence $\mathcal{B}$, and all of its superpatterns, are spectrally arbitrary.$\square$

Lemma 4.2.7

Proof Choose c, r, and l such that $\frac{n+1}{2} \leq c \leq n-1$, $2 \leq r \leq n-c+1$, and $l \notin \{1, c\}$. Let $\mathcal{C}$ be a subpattern of $\mathcal{W}_n$ where entry $c_{r,1} = 0$. Let $\mathcal{B}$ be the superpattern of $\mathcal{C}$ created by making the following entries nonzero:

- Entry $b_{n,c} = b_{n-c+1}$.
- Entry $b_{1,1} = a_1$.
- Entry $b_{c,c} = b_1$.
- Entry $b_{n,l} = y$.
- Entry $b_{c+r-1,c} = a_r$.

The following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - b_1$;

$f_k = -a_k + b_1 a_{k-1} + q_k$ for $2 \leq k \leq r-1$;

$f_r = -b_r + b_1 a_{r-1} + q_r$;

$f_{r+1} = -a_{r+1} + a_r a_1 + q_{r+1}$;

$f_k = -a_k + b_1 a_{k-1} + a_r a_{k-r} + q_k$ for $r+2 \leq k \leq 2r-1$;

$f_{2r} = -a_{2r} + b_1 a_{2r-2} + q_{2r}$;

$f_k = -a_k + b_1 a_{k-1} + a_r a_{k-r} + q_k$ for $2r+1 \leq k \leq n-c$;

$f_{n-c+1} = -a_{n-c+1} - b_{n-c+1} + b_1 a_{n-c} + a_r a_{n-c-r+1} + q_{n-c+1}$;

$f_k = -a_k + b_1 a_{k-1} + a_r a_{k-r} + b_{n-c+1} a_{k-n+c-1} + q_k$ for $n-c+2 \leq k \leq c$;

$f_k = -a_k + a_r a_{k-r} + b_{n-c+1} a_{k-n+c-1} + q_k$ for $c+1 \leq k \leq c+r-1$;

$f_k = -a_k + b_{n-c+1} a_{k-n+c-1} + q_k$ for $c+r \leq k \leq n$;

In each f_k, the q_k are the sum of the following terms where defined:

$-y$ for $k = n-l+1$;

$ya_{k-n+l-1}$ for $n-l+2 \leq k \leq n$ with $k \neq r$ if $l \geq 2$;

yb_1 for $k = n-l+2$ if $l \geq c+1$;

ya_r for $k = n-l+r+1$ if $l \geq c+r$;

$yb_1 a_{k-n+l-2}$ for $n-l+3 \leq k \leq n-l+c+1$ with $k \neq r$ if $l \geq c+1$;

$ya_r a_{k-n+l-r-1}$ for $n-l+r+3 \leq k \leq n-l+r+c$ with $k \neq r$ if $l \geq c+r$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, a_2, \ldots, a_n$ evaluated at any nilpotent realization is $(-1)^n$. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, this pattern and all of its superpatterns are spectrally arbitrary.

Solving for a_k in each of the equations $f_k = 0$ and applying back substitution, for $1 \leq k \leq n-c$ and $k \neq r$, a_k is a polynomial in b_1 of degree k and a_r is a polynomial in b_1 of degree r. Let $m = \lfloor \frac{c}{n-c+1} \rfloor$, then a_k is a polynomial in b_{n-c+1} of degree i for $k = i(n-c+1)+p$, $1 \leq i \leq m$, and $0 \leq p \leq n-c$, and of degree $m+1$ otherwise. Thus, we can choose b_1, b_{n-c+1} such that a nilpotent realization of $\mathcal{B}$ exists. Thus, $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.$\square$

Lemma 4.2.8

Proof Choose l and r such that $l \notin \{1, 3\}$ and $r > 4$. Let $\mathcal{C}$ be a subpattern of $\mathcal{W}_n$ where entry $c_{2,1} = 0$. Let $\mathcal{B}$ be the superpattern of $\mathcal{C}$ created by making the following entries nonzero:

- For all $k \notin \{1, 2, 3, 4, r\}$ $b_{k,2} = b_{k-1}$.
- Entry $b_{4,3} = a_2$.
- Entry $b_{1,1} = a_1$.
- Entry $b_{3,3} = b_1$.
- Entry $b_{n,l} = y$.

The following are the coefficient functions for the characteristic polynomial of $\mathcal{B}$:

$f_1 = -a_1 - b_1$;

$f_2 = -a_2 + a_1b_1 + q_2$;

$f_k = -a_k - b_k + a_1b_{k-1} + q_k$ for $3 \leq k \leq r - 1$;

$f_r = -a_r + a_1b_{r-1} + q_r$;

$f_{r+1} = -a_{r+1} - b_{r+1} + q_{r+1}$;

$f_k = -a_k - b_k + a_1b_{k-1} + q_k$ for $r + 2 \leq k \leq n - 1$;

$f_n = -a_n + q_n$;

In each f_k, the q_k are the sum of the following terms where defined:

$-y$ for $k = n - l + 1$;

a_1y for $k = n - l + 2$;

$a_{k-n+l-1}y$ for $n - l + 4 \leq k \leq n$ if $l \geq 4$;

$b_{k-n+l-1}y$ for $n - l + 3 \leq k \leq n - 2$ if $l \geq 5$ and for $k \neq n - l + r + 1$;

$-a_1 b_{k-n+l-2} y$ for $n-l+4 \leq k \leq n-1$ if ≥ 5 and for $k \neq n-l+r+2$;

The Jacobian of $f_1, f_2, ..., f_{n-1}, f_n$ with respect to the variables $a_1, a_2, a_3, \ldots, a_n$ evaluated at any nilpotent realization is $(-1)^n$. If we can guarantee a nilpotent realization for $\mathcal{B}$ exists, this pattern and all of its superpatterns are spectrally arbitrary.

Solving for a_k in each of the equations $f_k = 0$ and applying back substitution, $a_1 = -b_1$, $a_2 = -b_1^2$, and for $3 \leq k \leq n-2$, a_k is a linear function in b_k. Observe, a_{n-1} and a_n are both nonzero multiples of y. Thus, we can choose b_1, y, and b_k for $2 \leq k \leq n-1$ such that a nilpotent realization of $\mathcal{B}$ exists. Thus, $\mathcal{B}$ satisfies the Nilpotent-Jacobian condition. Hence, $\mathcal{B}$ and all of its superpatterns are spectrally arbitrary.$\square$

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