MULTIVARIATE COMPOUND POINT PROCESSES WITH DRIFTS

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WITH DRIFTS

Abstract

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This dissertation focuses on the multivariate compound point processes with drifts and their applications to reliability modeling and financial risk management.

Consider a system that consists of multiple components. The random shocks arrive at the system according to a stochastic point process, and each shock incurs several types of damages, one for each component, that are usually stochastically dependent. As long as the system is working, it generates some rewards, such as income, that can be used in maintenance of the system. The cumulative damages on various components over time can be described by a multivariate compound point process with drifts in which drift rates represent the reward rates. The performance measures of interests are various multivariate ruin/failure probabilities. These ruin/failure probabilities are of fundamental importance to the system operations and management, but there are no closed formulas even in some simplest cases.

We introduce in this dissertation a general multivariate compound point process with drifts, and discuss various ruin probabilities. Utilizing stochastic comparison methods, we analytically compare two such processes with different parameters and obtain some computable bounds for various ruin/failure probabilities that have no closed form expressions. We also apply the results to reliability and risk modeling and show how ignoring dependence among various types of damages would result in over-estimating or under-estimating the ruin/failure probabilities. In addition, We utilize multivariate phase-type

distributions to model random damages and obtain, via the matrix method, explicit expressions for certain ruin/failure probabilities. The results we obtained are illustrated by numerous examples and extensive stochastic simulations.

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Chapter 1

Introduction

This dissertation studies the multivariate compound point processes with drifts, and associated multivariate ruin probabilities. Utilizing the stochastic comparison methods, the dependence structure of a multivariate compound point process with drift is analyzed, and the computable bounds for the ruin probabilities, that are usually intractable, are obtained. The bounds for the ruin probabilities of multivariate process models with phase type distributed damage vectors are explicitly derived. The results obtained are illustrated by numerous examples and extensive stochastic simulations.

Section 1.1 discusses the motivation of our multivariate process models, and reviews the literature in this field. Section 1.2 outlines the organization of this dissertation, and summarizes our main results.

1.1 Motivation

The multivariate compound point processes with drifts have a wide variety of interpretations, but our study is mainly motivated from preventive maintenance for a system operating in a random environment. Consider a system that consists of a single component. The random shocks arrive at the system according to a stochastic point process, and each shock incurs some damage on the component. As long as the system is working, it generates some rewards, such as income, that can be used in maintenance of the system. The cumulative damages on the component over time can be described by a univariate point process with drift.

$$\sum_{n=1}^{N(t)} X_n - pt,\ t \geq 0, \tag{1.1.1}$$

where $\{N(t), t \geq 0\}$ is the shock arrival process, X_n is the damage on the component caused by the n-th shock, and p is the rate of the reward (drift rate). If the cumulative damage exceeds a certain threshold, say u, then the system fails. Thus the reliability measure of interest is the ruin or failure probability,

$$\psi(u) = P\left(\sum_{n=1}^{N(t)} X_n - pt > u \text{ for some } t > 0\right) = P\left(\sup_{0 \leq t < \infty}\left(\sum_{n=1}^{N(t)} X_n - pt\right) > u\right).$$

For example, if $N(t)$ is a Poisson process with rate λ, and the damage sizes X_n's are independent and identically distributed (i.i.d.) with common mean $1/\beta$, then one can calculate $\psi(u)$ explicitly,

$$\psi(u) = \frac{1}{1+\theta}\exp\left(-\frac{\theta\beta}{1+\theta}u\right), \tag{1.1.2}$$

where $\theta = p\beta/\lambda - 1 > 0$ is known as the *relative security loading* parameter. However, other than (1.1.2), the ruin probability $\psi(u)$ has no closed form expression in most cases.

The reliability applications of univariate compound point processes with drifts (1.1.1) can be found in Aven and Jensen (1999). Such processes also have extensive applications in financial risk management (Asmussen 2000). Consider an insurance or investment portfolio. The claim events occur according to a point process $\{N(t), t \geq 0\}$, and each event yields a claim with size $X_n \geq 0$. Then the claim surplus process of the portfolio is described by (1.1.1), where p is interpreted as the premium rate. The ruin probability $\psi(u)$ is of fundamental interest to preventive maintenance and risk management. Since the ruin probability is intractable in most cases, finding computable bounds for $\psi(u)$ is of significant importance in reliability modeling and risk management (Aven and Jensen 1999, Asmussen 2000).

Most practical systems consist of more than one component, and so the need to introduce multivariate compound point processes with drifts. Consider a system that consists of multiple components. The random shocks arrive at the system according to a stochastic point process $\{N(t), t \geq 0\}$, and each shock incurs several types of damages, one for each component, that are usually stochastically dependent. As long as the system is working, it generates some rewards, such as income, that can be used in maintenance of the system. The cumulative damage on various components over time can be described by a multivariate point process with drifts.

$$\boldsymbol{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_m(t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N(t)} X_{n,1} - p_1 t \\ \vdots \\ \sum_{n=1}^{N(t)} X_{n,m} - p_m t \end{pmatrix}, \quad t \geq 0, \tag{1.1.3}$$

where $(X_{n,1}, \ldots, X_{n,m})$ are the damages on various components caused by the n-th shock, and $(p_1, \ldots, p_m)$ is the vector of the reward (drift) rates. The performance measures of interest are various multivariate ruin/failure probabilities. For example, if one is interested in simultaneous failures, then the following ruin probability is of interest,

$$\psi_{sim}(u_1, \ldots, u_m) = P\left((S_1(t), \ldots, S_m(t)) > (u_1, \ldots, u_m) \text{ for some } t > 0\right),$$

where $(u_1, \ldots, u_m)$ is the vector of design thresholds. If one is interested in the aggregated damage accumulated on various components, then this ruin probability is of interest,

$$\psi_{sum}(u) = P\left(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S_j(t) > u\right).$$

The various ruin probabilities are of fundamental interest in preventive maintenance for multi-component systems and in multivariate risk analysis.

For the multivariate compound Poisson risk models, Sundt (1999) studied a recursive approach for the evaluation of the distribution of the multivariate cumulative process

$$\left\{\left(\sum_{n=1}^{N(t)} X_{n,1}, ..., \sum_{n=1}^{N(t)} X_{n,m}\right), t \geq 0\right\}.$$

Li and Xu (2001, 2002) studied the first passage times that this multivariate cumulative process exceeds various threshold values. Chan et al. (2003) discussed the ruin probability of the aggregate claim for the case where the claim sizes $X_{n,1}, ..., X_{n,m}$ are independent for any $n \geq 1$. Cai and Li (2005a) established the lower bound of certain ruin probabilities for the positively associated claims, and obtained an explicit expression of the ruin probability for the aggregate claim in a multivariate compound Poisson risk model whose claims of various types follow a so called multivariate phase type distribution. In general, however, the properties and expressions of the multivariate ruin probabilities are largely unknown. The goal of this dissertation is to present a systematic study on multivariate compound point processes with drifts and associated multivariate ruin probabilities, and to derive computable bounds for the multivariate ruin probabilities whose expressions are intractable.

1.2 Main Results and Organization

This dissertation is organized as follows.

1. In Chapter 2, we introduce multivariate compound point processes with drifts, and define several multivariate ruin probabilities. We also discuss the relations of our models with other models in the literature.

2. In Chapter 3, we obtain the stochastic comparison results for multivariate compound point processes with drifts. We show that if the shocks arrive more frequently in some *stochastic sense*, and the damages are larger *stochastically*, then the ruin probabilities become larger. We also obtain the bounds as by-products of our comparison results.

3. In Chapter 4, we present a dependence analysis for the multivariate ruin probabilities. We show that if the damage vectors are more dependent in some sense, then the ruin probabilities of common failures and the ruin probability of the aggregated damage become larger, whereas the ruin probability for at least one failure becomes smaller. This further illustrates that ignoring dependence among the components often results in over-estimating or under-estimating the ruin probabilities.

4. In Chapter 5, we derive various computable bounds for the multivariate ruin probabilities. Two approaches are used. First, we derive the bounds using univariate compound point processes, and second approach utilizes the positive dependence concepts. All the bounds discussed in this chapter depend on the dependence structure of damage vectors, and thus the effect of dependence on these bounds are also discussed.

   To facilitate the computations of these bounds, we utilize in this chapter multivariate phase type distributions. The multivariate phase type distributions include many well-known distributions, such as Marshall-Olkin distributions, and can be also used to approximate any multivariate life distributions. The explicit computations of the bounds for bivariate and trivariate Marshall-Olkin distributions are obtained, and extensive simulation results are also presented to illustrate the results.

Throughout this paper, the term ‘increasing’ and ‘decreasing’ mean ‘non-decreasing’ and ‘non-increasing’ respectively, and the measurability of sets and functions as well as the existence of expectations are often assumed without explicit mention. Any inequality between two vectors with finite or infinite dimensions means the inequalities component-wise. A product space of partially ordered sets is equipped with the component-wise partial ordering. All the random variables are denoted by capital letters, and random vectors are written in bold face. Any product of matrices and/or vectors are understood as a product with appropriate sizes.

Chapter 2

Multivariate Compound Point Processes with Drifts

This chapter introduces the multivariate compound point processes with drifts, and various ruin probabilities of interest to our study. The relations of the multivariate process models and univariate process models are discussed, and two applications from preventive maintenance and risk management are highlighted.

2.1 Multivariate Cumulative Processes

Consider a system that consists of m components. Random shocks arrive at the system according to a point process $\{\tau_n, n \geq 1\}$, and each shock incurs several types of damages, one for each component, that are usually stochastically dependent. Let $N(t)$ denote the number of shocks occurred prior to time $t > 0$, that is,

$$N(t) = \max\{n : \tau_n \leq t\}. \tag{2.1.1}$$

Let $X_{n,j}$ be the damage size of the n-th shock incurred on the j-th component, $1 \leq j \leq m$, $n \geq 1$, $\lambda_j(n)$ the recovery (drift) rate function of the j-th component when the n-th shock hits the system, $1 \leq j \leq m$, $n \geq 1$. The multivariate compound point process of the m components with drifts is described by

$$\boldsymbol{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_m(t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N(t)} X_{n,1} - \int_0^t \lambda_1(N(s))ds \\ \vdots \\ \sum_{n=1}^{N(t)} X_{n,m} - \int_0^t \lambda_m(N(s))ds \end{pmatrix}, \quad t \geq 0, \tag{2.1.2}$$

We assume throughout that $\{(X_{n,1}, \ldots, X_{n,m}), n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) non-negative random vectors, which is also independent of $\{N(t), t \geq 0\}$, but allow $X_{n,1}, \ldots, X_{n,m}$ to be dependent. We also assume that $\{N(t), t \geq 0\}$ is non-explosive, that is, that for any fixed $t > 0$, $N(t)$ is finite almost surely.

The multivariate stochastic process described in (2.1.2) has a variety of interpretations, but we are mainly interested in the following two applications.

1. *Maintenance and Repair Models*: Consider a system subjected to random shocks that occur according to a point process $\{\tau_n, n \geq 1\}$. In response to the damages incurred by random shocks, corrective repairs are carried out on all components, and take only a negligible amount of time. Let $X_{n,j}$ be the cost associated with the n-th repair on the j-th component, $1 \leq j \leq m$, $n \geq 1$, then the cumulative maintenance cost at the j-th component by time t is given by

$$\sum_{n=1}^{N(t)} X_{n,j}, \quad 1 \leq j \leq m.$$

   As long as the system is working, it generates income with the rate function $\lambda(n)$ that depends on the number n of shocks received. Then the cumulative net costs at various components by time t are given by

$$\boldsymbol{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_m(t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N(t)} X_{n,1} - \theta_1 \int_0^t \lambda(N(s))ds \\ \vdots \\ \sum_{n=1}^{N(t)} X_{n,m} - \theta_m \int_0^t \lambda(N(s))ds \end{pmatrix}, \quad t \geq 0, \qquad (2.1.3)$$

   where $\theta_1 \geq 0, ..., \theta_m \geq 0$ are the allocation parameters with $\sum_{j=1}^{m} \theta_j = 1$.

2. *Insurance Portfolios*: Consider an insurance or investment portfolio that consists of m sub-portfolios. The claim events occur according to a point process, and each event yields several types of claims, one for each sub-portfolio, that are usually stochastically dependent. Let $N(t)$ denote the number of claim events by time $t > 0$, and $X_{n,j}$ the type j claim size of the n-th event, $1 \leq j \leq m$, $n \geq 1$. The multivariate claim surplus process of the m sub-portfolios is described by

$$\boldsymbol{S}(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_m(t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N(t)} X_{n,1} - p_1 t \\ \vdots \\ \sum_{n=1}^{N(t)} X_{n,m} - p_m t \end{pmatrix}, \quad t \geq 0, \qquad (2.1.4)$$

where $p_j > 0$ is the premium rate in sub-portfolio j or for type j claim, $j = 1, ..., m$.

The system performance measures that we are interested in are various kinds of ruin probabilities. Let $u_j \geq 0$, $j = 1, ..., m$, denote the initial capital for component j in such a multivariate compound process model (2.1.2). A ruin event occurs if the cumulative costs/damages of some components exceed, in a certain fashion, their corresponding initial capital reserves. Various ruin probabilities in multivariate compound process models are of fundamental interest in maintenance and risk management. For example, consider the following four ruin probabilities.

$$\psi_{sum}(u) = P\left(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S_j(t) > u\right), \tag{2.1.5}$$

$$\psi_{and}(u_1, \ldots, u_m) = P\left(\bigcap_{j=1}^{m} \left\{\sup_{0 \leq t < \infty} (S_j(t)) > u_j\right\}\right), \tag{2.1.6}$$

$$\begin{aligned} \psi_{or}(u_1, \ldots, u_m) &= P\left(\bigcup_{j=1}^{m} \left\{\sup_{0 \leq t < \infty} (S_j(t)) > u_j\right\}\right) \qquad (2.1.7) \\ &= P\left(\sup_{0 \leq t < \infty} (\max\{S_1(t) - u_1, \ldots, S_m(t) - u_m\}) > 0\right), \end{aligned}$$

$$\begin{aligned} \psi_{sim}(u_1, \ldots, u_m) &= P\left((S_1(t), \ldots, S_m(t)) > (u_1, \ldots, u_m) \text{ for some } t > 0\right) \qquad (2.1.8) \\ &= P\left(\sup_{0 \leq t < \infty} (\min\{S_1(t) - u_1, \ldots, S_m(t) - u_m\}) > 0\right). \end{aligned}$$

The ruin probability in (2.1.5) denotes the probability that ruin occurs when the aggregate damage of all components exceeds a threshold. The ruin probability in (2.1.6) denotes the probability that ruin occurs, not necessarily at the same time, in all components eventually, whereas the ruin probability in (2.1.8) denotes the probability that ruin occurs in all components simultaneously or at the same instant in time. The ruin probability in (2.1.7) represents the probability that ruin occurs in at least one component. The focus of this dissertation is on these ruin probabilities for the multivariate compound point process models.

Example 2.1.1. In the univariate case that $m = 1$, we have

$$\psi(u) = \psi_{sum}(u) = \psi_{and}(u) = \psi_{or}(u) = \psi_{sim}(u) = P\left(\sup_{0\le t<\infty} S_1(t) > u\right), \tag{2.1.9}$$

where $u \geq 0$ is the initial capital. That is, all the ruin probabilities in (2.1.5), (2.1.6), (2.1.7), and (2.1.8) are the same. In the bivariate case, we have

$$\psi_{sum}(u) = P\left(\sup_{0\le t<\infty} (S_1(t) + S_2(t)) > u\right),$$

$$\psi_{and}(u_1, u_2) = P\left(\sup_{0\le t<\infty} (S_1(t)) > u_1,\ \sup_{0\le t<\infty} (S_2(t)) > u_2\right),$$

$$\psi_{or}(u_1, u_2) = P\left(\sup_{0\le t<\infty} (S_1(t)) > u_1,\ \text{or}\ \sup_{0\le t<\infty} (S_2(t)) > u_2\right),$$

$$\psi_{sim}(u_1, u_2) = P\left(\sup_{0\le t<\infty} (\min\{S_1(t) - u_1, S_2(t) - u_2\}) > 0\right).$$

□

2.2 Processes with Phase-Type Distributed Damages

In general, these ruin probabilities are intractable. As a matter of fact, even in the univariate case that $m = 1$, it is often difficult to obtain the explicit formula for its ruin probability $\psi(u)$. For the univariate model with the constant drift rate p, the Poisson arrival process with rate λ and the exponential damage sizes with common mean $1/\beta$, one can calculate $\psi(u)$ explicitly,

$$\psi(u) = \frac{1}{1+\theta} \exp\left(-\frac{\theta\beta}{1+\theta} u\right), \tag{2.2.1}$$

where $\theta = p\beta/\lambda - 1 > 0$ is known as the *relative security loading* parameter.

A well-known general result in the univariate case is due to Asmussen and Rolski (1991) who gave an explicit formula of $\psi(u)$ for the compound Poisson process model when the drift rate is p, the counting process $N(t)$ is Poisson with rate λ, and the damage size is of phase type in the sense of Neuts (1981). A non-negative random variable X is said to be of phase type with representation $(\boldsymbol{\alpha}, T, d)$ if X is the time to absorption into the absorbing state 0 in a finite Markov chain with state space $\{0, 1, \ldots, d\}$ and initial distribution $(0, \boldsymbol{\alpha})$, and infinitesimal generator,

$$\begin{bmatrix} 0 & \mathbf{0} \\ -T\boldsymbol{e} & T \end{bmatrix},$$

where $\mathbf{0}$ is the row vector of zeros of d dimension, and $\boldsymbol{e}$ is the column vector of 1's, and T is a $d \times d$ non-singular matrix. For a compound Poisson process model with the relative security loading parameter $\theta = \frac{p}{E(X_{n,1})\lambda} - 1 > 0$, if the damage size is of phase type with representation $(\boldsymbol{\alpha}, T, d)$, then $\psi(u)$ in (2.1.9) is the tail probability of the stationary waiting time in the $M/PH/1$ queue. Utilizing this fact, Asmussen and Rolski (1991) showed that for any $u \geq 0$,

$$\psi(u) = -\frac{\lambda}{p}\boldsymbol{\alpha} T^{-1} \exp\left\{\left(T - \frac{\lambda}{p}\boldsymbol{t}_0\boldsymbol{\alpha} T^{-1}\right) u\right\} \boldsymbol{e}, \tag{2.2.2}$$

where $\boldsymbol{t}_0 = -T\boldsymbol{e}$. The phase type distributions enjoy many desirable properties (Neuts 1981), and in particular, any distribution on $[0, \infty)$ can be approximated by phase type distributions. Thus (2.2.2) is versatile in applications.

Ruin theory for the univariate model has been discussed extensively in the literature, and many results are summarized in Asmussen (2000) and Rolski et al. (1999). For the multivariate compound Poisson risk models, Sundt (1999) studied a recursive approach for the evaluation of the distribution of the multivariate cumulative process

$$\left\{\left(\sum_{n=1}^{N(t)} X_{n,1}, ..., \sum_{n=1}^{N(t)} X_{n,m}\right), t \geq 0\right\}.$$

Li and Xu (2001, 2002) studied the first passage times that this multivariate cumulative process exceeds various threshold values. Chan et al. (2003) discussed the ruin probability of the aggregate claim, $\psi_{or}(u_1, \ldots, u_m)$ and $\psi_{sim}(u_1, \ldots, u_m)$ for the case where the claim sizes $X_{n,1}, ..., X_{n,m}$ are independent for any $n \geq 1$. Cai and Li (2005a) established the lower bound of $\psi_{and}(u_1, \ldots, u_m)$ for the positively associated claims, and obtained an explicit expression of the ruin probability for the aggregate claim in a multivariate compound Poisson risk model whose claims of various types follow a multivariate phase type distribution. In general, however, the properties and expressions of the multivariate ruin probabilities are largely unknown. In this dissertation, we investigate the dependence properties of the ruin probabilities (2.1.5)-(2.1.8), and establish the sharp upper and lower bounds of (2.1.5)-(2.1.8) whose explicit expressions are intractable even in the simplest cases, such as multivariate compound Poisson process models with multivariate exponentially distributed claims.

Chapter 3

Stochastic Comparisons of Multivariate Compound Point Processes with Drifts

This chapter compares stochastically two multivariate compound point processes with drifts, and studies how changes in the shock arrival process and in damage size vectors would affect various ruin probabilities. The idea is to order the shock arrival processes and damage size vectors of two multivariate process models in some ways, and then utilize the preservation properties of stochastic orders and model structures to establish the comparison results. The comparison methods of random vectors and of point processes are reviewed, the comparisons of ruin probabilities for the shock arrival processes and damage vectors that are stochastically ordered are established. Various examples are presented to illustrate the results.

3.1 Stochastic Comparisons

There are various ways of comparing stochastically two random vectors. The following notions of stochastically comparisons can be found in Shaked and Shanthikumar (1994), and Müller and Stoyan (2002).

Definition 3.1.1. Let $\boldsymbol{X} = (X_1, \ldots, X_m)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ be two $\mathcal{R}^m$-valued random vectors. $\boldsymbol{X}$ is said to be larger than $\boldsymbol{Y}$ in stochastic order, denoted by $\boldsymbol{X} \geq_{st} \boldsymbol{Y}$, if $Ef(\boldsymbol{X}) \geq Ef(\boldsymbol{Y})$ for all increasing functions f.

If $\boldsymbol{X} \geq_{st} \boldsymbol{Y}$ and $\boldsymbol{Y} \geq_{st} \boldsymbol{X}$, then $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same distribution, which will be denoted by $\boldsymbol{X} =_{st} \boldsymbol{Y}$ in this dissertation. It can be shown (see, for example, Shaked and Shanthikumar 1994) that $\boldsymbol{X} \geq_{st} \boldsymbol{Y}$ if and only if there exist two random vectors $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ defined on the same probability space such that

$$\boldsymbol{X}' \geq \boldsymbol{Y}', \text{ almost surely,} \tag{3.1.1}$$

and $\boldsymbol{X}'$ and $\boldsymbol{X}$ have the same distribution, $\boldsymbol{Y}'$ and $\boldsymbol{Y}$ have the same distribution.

Note that the stochastic order implies a variety of inequalities. For example, if $\boldsymbol{X} \geq_{st} \boldsymbol{Y}$, then it is easy to verify that

$$P(X_1 > x_1, \ldots, X_m > x_m) \geq P(Y_1 > x_1, \ldots, Y_m > x_m), \tag{3.1.2}$$

$$P(X_1 \leq x_1, \ldots, X_m \leq x_m) \leq P(Y_1 \leq x_1, \ldots, Y_m \leq x_m), \tag{3.1.3}$$

for any $(x_1, \ldots, x_m)$. If, in addition, both $\boldsymbol{X}$ and $\boldsymbol{Y}$ are non-negative, then we have

$$E(X_1^{i_1} \ldots X_m^{i_m}) \geq E(Y_1^{i_1} \ldots Y_m^{i_m}),$$

for any $i_1 \geq 0, \ldots, i_m \geq 0$. The stochastic order is closed under increasing transformations, convolutions and marginalizations.

For the univariate case, the stochastic order reduces the comparison of survival functions. Let X have the distribution function $F(x)$ and Y have the distribution function $G(x)$. Then $X \leq_{st} Y$ means that $\bar{F}(x) \leq \bar{G}(x)$, where $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$, or equivalently, $F(x) \geq G(x)$. For example, let X have the survival function $\bar{F}(x) = e^{-\lambda x}$ and Y have the survival function $\bar{G}(x) = e^{-\mu x}$, then if $\lambda > \mu$ we have $X \leq_{st} Y$.

To compare stochastically two point processes, we express any point process in terms of its counting process. A point process on $\mathcal{R}_+$ can be described as a sequence of random variables $0 = \tau_0 < \tau_1 < \cdots$ on a common probability space. We assume that $\lim_{n\to\infty} \tau_n = \infty$ almost surely; that is, that the process is non-explosive. An alternative description of the point process $\{\tau_n\}$ is through its associated counting process

$$N(t) = \max\{n : \tau_n \leq t\}, \quad t \geq 0. \tag{3.1.4}$$

Thus, the process $N = \{N(t), t \geq 0\}$ can be viewed as a random element of $\mathcal{D}[0, \infty)$ (the space of real functions on $[0, \infty)$ which are right-continuous with left-hand limits). As is mentioned in Shaked and Szekli (1995), (3.1.4) provides us with a "time-dynamic" view of the point process. Note that $\mathcal{D}[0,\infty)$ is a Polish space with the partial order defined, for $f, g \in \mathcal{D}[0,\infty)$, by

$$f \leq_{\mathcal{D}} g, \quad \text{if} \quad f(t) \leq g(t),\ t \geq 0.$$

The description above yields the following type of stochastic comparison of point processes.

Definition 3.1.2. Suppose that $N = \{N(t)\}$ and $N' = \{N'(t)\}$ are two counting processes. Define $N \leq_{\text{st-}\mathcal{D}} N'$ if $E\phi(\{N(t)\}) \leq E\phi(\{N'(t)\})$ for all $\leq_{\mathcal{D}}$-increasing functions ϕ on $\mathcal{D}[0,\infty)$.

Note that $N \leq_{\text{st-}\mathcal{D}} N'$ implies stochastic comparisons of finite dimensional distributions of point processes. For example, if $N \leq_{\text{st-}\mathcal{D}} N'$, then $N(t) \leq_{st} N'(t)$ for any fixed t.

It follows from Szekli (1995) that $N \leq_{\text{st-}\mathcal{D}} N'$ if and only if one can construct two counting processes

$$M = \{M(t), t \geq 0\} \text{ and } M' = \{(M'(t), t \geq 0\},$$

on the same probability space, such that M and N have the same distribution, and M' and N' have the same distribution, and

$$M(t) \leq M'(t),\ \text{for all } t \geq 0,\ \text{almost surely.} \tag{3.1.5}$$

If the jump times of N are $\tau = \{\tau_1, \tau_2, \dots,\}$, and the jump times of N' are $\tau' = \{\tau_1', \tau_2', \dots,\}$, then the stochastic comparison $N \leq_{\text{st-}\mathcal{D}} N'$ is equivalent to the comparison

$$E\phi(\tau) \geq E\phi(\tau'),$$

for all componentwise increasing functions ϕ on $\mathbb{R}_+^\infty$. Equivalently, we can construct two sequences of non-negative random variables

$$\boldsymbol{T} = (T_1, T_2, \dots) \text{ and } \boldsymbol{T}' = (T_1', T_2', \dots)$$

on the same probability space, such that $\boldsymbol{T}$ and τ have the same distribution, and $\boldsymbol{T}'$ and τ' have the same distribution, and

$$T_n \geq T_n' \text{ for all } n \geq 1. \tag{3.1.6}$$

The coupling ideas in (3.1.5) and (3.1.6) will be used in this dissertation to facilitate the proofs of the comparison results.

For renewal processes, the comparison can be easily established.

Lemma 3.1.3. Let $X \sim F(x)$ and $Y \sim G(x)$ denote the interarrival times of two renewal processes $N_F = \{N_F(t)\}$ and $N_G = \{N_G(t)\}$ respectively. If $X \leq_{st} Y$, then $N_F \geq_{\text{st-}\mathfrak{D}} N_G$.

Proof. Two counting processes $N_F(t)$ and $N_G(t)$ can be written as

$$N_F(t) = \max\{n : \sum_{i=1}^{n} X_i \leq t\},$$

$$N_G(t) = \max\{n : \sum_{i=1}^{n} Y_i \leq t\},$$

where X_i's are i.i.d. with distribution F, and Y_i's are i.i.d. with distribution G. Obviously $N_F(t)$ is decreasing in X_i, and $N_G(t)$ is decreasing in Y_i, for all $i \geq 1$. Since $X_i \leq_{st} Y_i$, then, from (3.1.1), we can constract X_i' and Y_i' on the same probability space such that

$$X_i' \leq Y_i, \text{ for all } i, \text{ almost surely},$$

and X_i' and X_i have the same distribution, and Y_i' and Y_i have the same distribution. Thus

$$\max\{n : \sum_{i=1}^{n} X_i' \leq t\} \geq \max\{n : \sum_{i=1}^{n} Y_i' \leq t\} \text{ for all } t,$$

almost surely. This implies that $N_F \geq_{\text{st-}\mathfrak{D}} N_G$. □

If N_F and N_G are two Poisson processes with rates λ and μ respectively, then Lemma 3.1.3 implies that if $\lambda \geq \mu$, then $N_F \geq_{\text{st-}\mathfrak{D}} N_G$. For Non-homogeneous Poisson Processes, the comparison can be also established (Szekli 1995).

Lemma 3.1.4. Let $m_1(t)$ and $m_2(t)$ be two mean value functions of two nonhomogeneous Poisson processes $N_1 = \{N_1(t)\}$ and $N_2 = \{N_2(t)\}$ respectively. If $m_1(t) \leq m_2(t)$, then $N_1 \leq_{\text{st-}\mathfrak{D}} N_2$.

Example 3.1.5. Let $m(t)$ be the mean value function of a nonhomogeneous Poisson process $N_{NHPP} = \{N_{NHPP}(t)\}$, and λ be the rate of a Poisson process $N_{PP} = \{N_{PP}(t)\}$. If $\lambda(t) \leq \frac{1}{\lambda}$, then $N_{NHPP} \leq_{\text{st-}\mathfrak{D}} N_{PP}$. □

3.2 Stochastic Comparisons of Multivariate Compound Point Processes with Drifts

Let $\boldsymbol{S}^1(t)$ and $\boldsymbol{S}^2(t)$ be two multivariate compound point processes with the same drift functions $\lambda_j(n)$, $1 \leq j \leq m$, the same damage sizes $(X_{n,1}, \ldots, X_{n,m})$, but different shock arrival processes $N_1 = \{N_1(t)\}$ and $N_2 = \{N_2(t)\}$, respectively. Let $\psi^i_{sum}(u)$ $(\psi^i_{or}(u_1, \ldots, u_m), \psi^i_{and}(u_1, \ldots, u_m), \psi^i_{sim}(u_1, \ldots, u_m))$ be the ruin probability of type (2.1.5) ((2.1.6), (2.1.7), (2.1.8)) for the process $\boldsymbol{S}^i(t)$, $i = 1, 2$.

Theorem 3.2.1. Assume that the drift functions $\lambda_j(u)$ are decreasing in u. If $N_1 \geq_{\text{st-}\mathfrak{D}} N_2$, then

1. $\psi^1_{sum}(u) \geq \psi^2_{sum}(u)$,
2. $\psi^1_{and}(u_1, \ldots, u_m) \geq \psi^2_{and}(u_1, \ldots, u_m)$,
3. $\psi^1_{or}(u_1, \ldots, u_m) \geq \psi^2_{or}(u_1, \ldots, u_m)$, and
4. $\psi^1_{sim}(u_1, \ldots, u_m) \geq \psi^2_{sim}(u_1, \ldots, u_m)$.

Proof. (1) From (2.1.5), we have

$$\psi^1_{sum}(u) = P\left(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S^1_j(t) > u\right),$$

$$\psi^2_{sum}(u) = P\left(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S^2_j(t) > u\right),$$

where

$$S^1_j(t) = \sum_{n=1}^{N_1(t)} X_{n,j} - \int_0^t \lambda_j(N_1(s))ds, \quad t \geq 0,$$

$$S^2_j(t) = \sum_{n=1}^{N_2(t)} X_{n,j} - \int_0^t \lambda_j(N_2(s))ds, \quad t \geq 0,$$

$$N_1 = \{N_1(t), \quad t \geq 0\},$$

$$N_2 = \{N_2(t), \quad t \geq 0\}.$$

If $N_1 \geq_{\text{st-}\mathfrak{D}} N_2$, one can construct two counting processes on the same probability space

$$\tilde{N}_1 = \{\tilde{N}_1(t), \quad t \geq 0\},$$

$$\tilde{N}_2 = \{\tilde{N}_2(t), \quad t \geq 0\},$$

satisfying two conditions:

1. $\tilde{N}_1(t) \geq \tilde{N}_2(t), \quad \forall t \geq 0,$

2. $\tilde{N}_1$ and N_1 have the same distribution, $\tilde{N}_2$ and N_2 have the same distribution.

Consider,

$$\tilde{S}_j^1(t) = \sum_{n=1}^{\tilde{N}_1(t)} X_{n,j} - \int_0^t \lambda_j(\tilde{N}_1(s))ds, \quad t \geq 0,$$

$$\tilde{S}_j^2(t) = \sum_{n=1}^{\tilde{N}_2(t)} X_{n,j} - \int_0^t \lambda_j(\tilde{N}_2(s))ds, \quad t \geq 0.$$

Since $\lambda_j(u)$ is decreasing in u, then $\tilde{N}_1 \geq \tilde{N}_2$ implies $\int_0^t \lambda_j(\tilde{N}_1(s))ds \leq \int_0^t \lambda_j(\tilde{N}_2(s))ds$. Thus we have

$$\tilde{S}_j^1(t) \geq \tilde{S}_j^2(t), \quad \forall j = 1, 2, \ldots, m. \tag{3.2.1}$$

This sample path comparison implies that

$$\sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^1(t) \geq \sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^2(t) \Rightarrow P(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^1(t) > u) \geq P(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^2(t) > u).$$

Since

$$\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S_j^1(t) =_{st} \sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^1(t)$$

$$\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S_j^2(t) =_{st} \sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^2(t)$$

we have, $\psi_{sum}^1(u) \geq \psi_{sum}^2(u)$.

(2) From (2.1.6) and the sample path construction in (1),

$$\psi_{and}^1(u_1, \ldots, u_m) = P\left(\bigcap_{j=1}^{m} \left\{\sup_{0 \leq t < \infty} (\tilde{S}_j^1(t)) > u_j\right\}\right),$$

$$\psi_{and}^2(u_1, \ldots, u_m) = P\left(\bigcap_{j=1}^{m} \left\{\sup_{0 \leq t < \infty} (\tilde{S}_j^2(t)) > u_j\right\}\right),,$$

where

$$\tilde{S}_j^1(t) = \sum_{n=1}^{\tilde{N}_1(t)} X_{n,j} - \int_0^t \lambda_j(\tilde{N}_1(s))ds, \quad t \geq 0,$$

$$\tilde{S}_j^2(t) = \sum_{n=1}^{\tilde{N}_2(t)} X_{n,j} - \int_0^t \lambda_j(\tilde{N}_2(s))ds, \quad t \geq 0.$$

It follows from (3.2.1) that for each j

$$\left\{ \sup_{0 \leq t < \infty} \tilde{S}_j^2(t) > u_j \right\} \subseteq \left\{ \sup_{0 \leq t < \infty} \tilde{S}_j^1(t) > u_j \right\},$$

which implies that

$$P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}_j^1(t)) > u_j \right\} \right) \geq P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}_j^2(t)) > u_j \right\} \right).$$

Therefore, $\psi_{and}^1(u_1, \ldots, u_m) \geq \psi_{and}^2(u_1, \ldots, u_m)$.

(3) From (2.1.7) and the sample path construction in (1),

$$\psi_{or}^1(u_1, \ldots, u_m) = P\left(\bigcup_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (S_j^1(t)) > u_j \right\} \right),$$

$$\psi_{or}^2(u_1, \ldots, u_m) = P\left(\bigcup_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (S_j^2(t)) > u_j \right\} \right),,$$

where

$$\tilde{S}_j^1(t) = \sum_{n=1}^{\tilde{N}_1(t)} X_{n,j} - \int_0^t \lambda_j(\tilde{N}_1(s))ds, \quad t \geq 0,$$

$$\tilde{S}_j^2(t) = \sum_{n=1}^{\tilde{N}_2(t)} X_{n,j} - \int_0^t \lambda_j(\tilde{N}_2(s))ds, \quad t \geq 0.$$

It follows from (3.2.1) that for each j

$$\left\{ \sup_{0 \leq t < \infty} \tilde{S}_j^2(t) > u_j \right\} \subseteq \left\{ \sup_{0 \leq t < \infty} \tilde{S}_j^1(t) > u_j \right\},$$

which implies that

$$P\left(\bigcup_{j=1}^{m}\left\{\sup_{0\le t<\infty}(\tilde{S}_j^1(t))>u_j\right\}\right)\ge P\left(\bigcup_{j=1}^{m}\left\{\sup_{0\le t<\infty}(\tilde{S}_j^2(t))>u_j\right\}\right).$$

Therefore, $\psi_{or}^1(u_1,\ldots,u_m)\ge\psi_{or}^2(u_1,\ldots,u_m)$.

(4) From (2.1.8) and the sample path construction in (1),

$$\psi_{sim}^1(u_1,\ldots,u_m)=P\left((\tilde{S}_1^1(t),\ldots,\tilde{S}_m^1(t))>(u_1,\ldots,u_m)\text{ for some }t>0\right),$$

$$\psi_{sim}^2(u_1,\ldots,u_m)=P\left((\tilde{S}_1^2(t),\ldots,\tilde{S}_m^2(t))>(u_1,\ldots,u_m)\text{ for some }t>0\right),$$

where

$$\tilde{S}_j^1(t)=\sum_{n=1}^{\tilde{N}_1(t)}X_{n,j}-\int_0^t\lambda_j(\tilde{N}_1(s))ds,\quad t\ge 0,$$

$$\tilde{S}_j^2(t)=\sum_{n=1}^{\tilde{N}_2(t)}X_{n,j}-\int_0^t\lambda_j(\tilde{N}_2(s))ds,\quad t\ge 0.$$

It follows from (3.2.1) that

$$\{(\tilde{S}_1^2(t),\ldots,\tilde{S}_m^2(t))>(u_1,\ldots,u_m)\text{ for some }t>0\}$$

$$\subseteq\{(\tilde{S}_1^1(t),\ldots,\tilde{S}_m^1(t))>(u_1,\ldots,u_m)\text{ for some }t>0\},$$

which implies that

$$P\left((\tilde{S}_1^1(t),\ldots,\tilde{S}_m^1(t))>(u_1,\ldots,u_m)\text{ for some }t>0\right)$$

$$\ge P\left((\tilde{S}_1^2(t),\ldots,\tilde{S}_m^2(t))>(u_1,\ldots,u_m)\text{ for some }t>0\right).$$

Therefore, $\psi_{sim}^1(u_1,\ldots,u_m)\ge\psi_{sim}^2(u_1,\ldots,u_m)$. □

Example 3.2.2. Consider the ruin probability $\psi_{sum}(u)$ in (2.1.5) where the shock arrival process $N=\{N(t),t\ge 0\}$ is a nonhomogenrous Poisson process with rate function $\lambda(t)$, and independent and exponentially distributed damage sizes $(X_{n,1},\ldots,X_{n,m})$ with mean μ. The ruin probability $\psi_{sum}(u)$ can not be calculated explicitly. However, if $\lambda(t)$ is bounded, then the computable bounds for $\psi_{sum}(u)$ can be derived using Theorem 3.2.1.

Suppose that $a \leq \lambda(t) \leq b$. It follows from Example 3.1.5 that

$$N_a \geq_{\text{st-}\mathcal{D}} N \geq_{\text{st-}\mathcal{D}} N_b,$$

where N_a and N_b are two Poisson processes with rates a and b respectively. From Theorem 3.2.1, we have

$$\psi^a_{sum}(u) \geq \psi_{sum}(u) \geq \psi^b_{sum}(u),$$

where $\psi^a_{sum}(u)$ and $\psi^b_{sum}(u)$ are the ruin probabilities of type (2.1.5) with the same drift functions, same damage sizes as that of $\psi_{sum}(u)$, and Poisson shock arrival processes N_a and N_b respectively. □

Theorem 3.2.1 illustrates how change in a shock arrival process would affect various ruin probabilities. To understand the effect of damage sizes on the ruin probabilities, consider two multivariate compound point processes $\boldsymbol{S}^1(t)$ and $\boldsymbol{S}^2(t)$ with the same drift functions, the same shock arrival process, but different damage sizes $(X_{n,1}, \ldots, X_{n,m})$ and $(Y_{n,1}, \ldots, Y_{n,m})$, respectively. Again, let $\psi^i_{sum}(u)$ ($\psi^i_{or}(u_1, \ldots, u_m)$, $\psi^i_{and}(u_1, \ldots, u_m)$, $\psi^i_{sim}(u_1, \ldots, u_m)$) be the ruin probability of type (2.1.5) ((2.1.6), (2.1.7), (2.1.8)) for the process $\boldsymbol{S}^i(t)$, $i = 1, 2$.

Theorem 3.2.3. If $(X_{n,1}, \ldots, X_{n,m}) \geq_{st} (Y_{n,1}, \ldots, Y_{n,m})$ then

1. $\psi^1_{sum}(u) \geq \psi^2_{sum}(u)$,
2. $\psi^1_{and}(u_1, \ldots, u_m) \geq \psi^2_{and}(u_1, \ldots, u_m)$,
3. $\psi^1_{or}(u_1, \ldots, u_m) \geq \psi^2_{or}(u_1, \ldots, u_m)$, and
4. $\psi^1_{sim}(u_1, \ldots, u_m) \geq \psi^2_{sim}(u_1, \ldots, u_m)$.

Proof. (1) From (2.1.5), we have

$$\psi^1_{sum}(u) = P\left(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S^1_j(t) > u\right),$$

$$\psi^2_{sum}(u) = P\left(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S^2_j(t) > u\right),$$

where

$$S^1_j(t) = \sum_{n=1}^{N(t)} X_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0,$$

$$S_j^2(t) = \sum_{n=1}^{N(t)} Y_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0,$$

If $(X_{n,1}, \ldots, X_{n,m}) \geq_{st} (Y_{n,1}, \ldots, Y_{n,m})$, one can construct random vectors on the same probability space

$$(\tilde{X}_{n,1}, \ldots, \tilde{X}_{n,m}),\ (\tilde{Y}_{n,1}, \ldots, \tilde{Y}_{n,m}),\ n \geq 1$$

satisfying two conditions:

1. $(\tilde{X}_{n,1}, \ldots, \tilde{X}_{n,m}) \geq (\tilde{Y}_{n,1}, \ldots, \tilde{Y}_{n,m})$, almost surely, for all $n \geq 1$,
2. $(\tilde{X}_{n,1}, \ldots, \tilde{X}_{n,m})$ and $(X_{n,1}, \ldots, X_{n,m})$ have the same distribution, $(\tilde{Y}_{n,1}, \ldots, \tilde{Y}_{n,m})$ and $(Y_{n,1}, \ldots, Y_{n,m})$ have the same distribution.

Consider,

$$\tilde{S}_j^1(t) = \sum_{n=1}^{N(t)} \tilde{X}_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0,$$

$$\tilde{S}_j^2(t) = \sum_{n=1}^{N(t)} \tilde{Y}_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0.$$

Since $(\tilde{X}_{n,1}, \ldots, \tilde{X}_{n,m}) \geq (\tilde{Y}_{n,1}, \ldots, \tilde{Y}_{n,m})$ for all $n \geq 1$, we have

$$\tilde{S}_j^1(t) \geq \tilde{S}_j^2(t), \quad \forall j = 1, 2, \ldots, m. \tag{3.2.2}$$

This sample path comparison implies that

$$\sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^1(t) \geq \sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^2(t) \Rightarrow P(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^1(t) > u) \geq P(\sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^2(t) > u).$$

Since

$$\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S_j^1(t) =_{st} \sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^1(t)$$

$$\sup_{0 \leq t < \infty} \sum_{j=1}^{m} S_j^2(t) =_{st} \sup_{0 \leq t < \infty} \sum_{j=1}^{m} \tilde{S}_j^2(t)$$

we have, $\psi^1_{sum}(u) \geq \psi^2_{sum}(u)$.

(2) From (2.1.6) and the sample path construction in (1),

$$\psi^1_{and}(u_1, \ldots, u_m) = P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}^1_j(t)) > u_j \right\}\right),$$

$$\psi^2_{and}(u_1, \ldots, u_m) = P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}^2_j(t)) > u_j \right\}\right),,$$

where

$$\tilde{S}^1_j(t) = \sum_{n=1}^{N(t)} \tilde{X}_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0,$$

$$\tilde{S}^2_j(t) = \sum_{n=1}^{N(t)} \tilde{Y}_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0.$$

It follows from (3.2.2) that for each j

$$\left\{ \sup_{0 \leq t < \infty} \tilde{S}^2_j(t) > u_j \right\} \subseteq \left\{ \sup_{0 \leq t < \infty} \tilde{S}^1_j(t) > u_j \right\},$$

which implies that

$$P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}^1_j(t)) > u_j \right\}\right) \geq P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}^2_j(t)) > u_j \right\}\right).$$

Therefore, $\psi^1_{and}(u_1, \ldots, u_m) \geq \psi^2_{and}(u_1, \ldots, u_m)$.

(3) From (2.1.7) and the sample path construction in (1),

$$\psi^1_{or}(u_1, \ldots, u_m) = P\left(\bigcup_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (S^1_j(t)) > u_j \right\}\right),$$

$$\psi^2_{or}(u_1, \ldots, u_m) = P\left(\bigcup_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (S^2_j(t)) > u_j \right\}\right),,$$

where

$$\tilde{S}_j^1(t) = \sum_{n=1}^{N(t)} \tilde{X}_{n,j} - \lambda_j(N(t))t, \quad t \geq 0,$$

$$\tilde{S}_j^2(t) = \sum_{n=1}^{N(t)} \tilde{Y}_{n,j} - \lambda_j(N(t))t, \quad t \geq 0.$$

It follows from (3.2.2) that for each j

$$\left\{ \sup_{0 \leq t < \infty} \tilde{S}_j^2(t) > u_j \right\} \subseteq \left\{ \sup_{0 \leq t < \infty} \tilde{S}_j^1(t) > u_j \right\},$$

which implies that

$$P\left(\bigcup_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}_j^1(t)) > u_j \right\} \right) \geq P\left(\bigcup_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} (\tilde{S}_j^2(t)) > u_j \right\} \right).$$

Therefore, $\psi_{or}^1(u_1, \ldots, u_m) \geq \psi_{or}^2(u_1, \ldots, u_m)$.

(4) From (2.1.8) and the sample path construction in (1),

$$\psi_{sim}^1(u_1, \ldots, u_m) = P\left((\tilde{S}_1^1(t), \ldots, \tilde{S}_m^1(t)) > (u_1, \ldots, u_m) \text{ for some } t > 0 \right),$$

$$\psi_{sim}^2(u_1, \ldots, u_m) = P\left((\tilde{S}_1^2(t), \ldots, \tilde{S}_m^2(t)) > (u_1, \ldots, u_m) \text{ for some } t > 0 \right),$$

where

$$\tilde{S}_j^1(t) = \sum_{n=1}^{N(t)} \tilde{X}_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0,$$

$$\tilde{S}_j^2(t) = \sum_{n=1}^{N(t)} \tilde{Y}_{n,j} - \int_0^t \lambda_j(N(s))ds, \quad t \geq 0.$$

It follows from (3.2.2) that

$$\{(\tilde{S}_1^2(t), \ldots, \tilde{S}_m^2(t)) > (u_1, \ldots, u_m) \text{ for some } t > 0\}$$

$$\subseteq \{(\tilde{S}_1^1(t), \ldots, \tilde{S}_m^1(t)) > (u_1, \ldots, u_m) \text{ for some } t > 0\},$$

which implies that

$$P\left((\tilde{S}_1^1(t),\ldots,\tilde{S}_m^1(t)) > (u_1,\ldots,u_m) \text{ for some } t>0\right)$$

$$\geq P\left((\tilde{S}_1^2(t),\ldots,\tilde{S}_m^2(t)) > (u_1,\ldots,u_m) \text{ for some } t>0\right).$$

Therefore, $\psi_{sim}^1(u_1,\ldots,u_m) \geq \psi_{sim}^2(u_1,\ldots,u_m)$. □

Chapter 4

Dependence of Multivariate Compound Point Processes with Drifts

This chapter focuses on the dependence comparison of two multivariate compound point processes with drifts. To achieve this, we fix the marginal distributions of damage size vectors, increase the dependence of damage size vectors in some sense, and study the effect on various ruin probabilities. We show that increasing dependence increases the ruin probabilities of common component failures.

The dependence comparison methods are reviewed, and examples are presented throughout to illustrate the results.

4.1 Dependence Comparison Methods

Definition 4.1.1. Let $\boldsymbol{X} = (X_1, \ldots, X_m)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ be two $\mathcal{R}^m$-valued random vectors.

1. $\boldsymbol{X}$ is said to be more upper-orthant dependent than $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \geq_{uod} \boldsymbol{Y}$, if X_i and Y_i have the same distribution for each i, and

$$P(X_1 > x_1, \ldots, X_m > x_m) \geq P(Y_1 > x_1, \ldots, Y_m > x_m).$$

2. $\boldsymbol{X}$ is said to be more lower-orthant dependent than $\boldsymbol{Y}$, denoted by $\boldsymbol{X} \geq_{lod} \boldsymbol{Y}$, if X_i

and Y_i have the same distribution for each i, and

$$P(X_1 \le x_1, \dots, X_m \le x_m) \ge P(Y_1 \le x_1, \dots, Y_m \le x_m).$$

3. $\boldsymbol{X}$ is said to be more dependent than $\boldsymbol{Y}$ in supermodular order, denoted by $\boldsymbol{X} \ge_{sm} \boldsymbol{Y}$, if $Ef(\boldsymbol{X}) \ge Ef(\boldsymbol{Y})$ for all supermodular functions f; that is, functions satisfying that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^m$,

$$f(\boldsymbol{x} \vee \boldsymbol{y}) + f(\boldsymbol{x} \wedge \boldsymbol{y}) \ge f(\boldsymbol{x}) + f(\boldsymbol{y}),$$

where $\boldsymbol{x} \vee \boldsymbol{y}$ denotes the vector of component-wise maximums, and $\boldsymbol{x} \wedge \boldsymbol{y}$ denotes the vector of component-wise minimums.

If f has second partial derivatives, then the supermodular property of function f is equivalent to

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \ge 0,$$

for all $i \ne j$. The examples of supermodular functions include the functions of the following form,

$$f(x_1, \dots, x_s) = \prod_{j=1}^{s} f_j(x_j), \tag{4.1.1}$$

where $f_1, \dots, f_s$ are monotone in the same directions.

These stochastic orders have many useful properties and applications, and are studied in details in Marshall and Olkin (1979), Shaked and Shanthikumar (1994), and Müller and Stoyan (2002), and references therein. The following properties are frequently used in this and the next sections.

Lemma 4.1.2. Let $\boldsymbol{X} = (X_1, \dots, X_m)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_m)$ be two $\mathcal{R}^m$-valued random vectors.

1. If $\boldsymbol{X} \ge_{uod} \boldsymbol{Y}$, then $(f_1(X_1), \dots, f_m(X_m)) \ge_{uod} (f_1(Y_1), \dots, f_m(Y_m))$ for any functions $f_1, \dots, f_m$ that are all increasing.

2. If $\boldsymbol{X} \ge_{lod} \boldsymbol{Y}$, then $(f_1(X_1), \dots, f_m(X_m)) \ge_{lod} (f_1(Y_1), \dots, f_m(Y_m))$ for any functions $f_1, \dots, f_m$ that are all increasing.

3. If $\boldsymbol{X} \ge_{sm} \boldsymbol{Y}$, then $(f_1(X_1), \dots, f_m(X_m)) \ge_{sm} (f_1(Y_1), \dots, f_m(Y_m))$ for any functions $f_1, \dots, f_m$ that are all increasing or all decreasing.

4. If $\boldsymbol{X} \geq_{sm} \boldsymbol{Y}$, then X_j and Y_j have the same marginal distribution for any $j = 1, \ldots, m$, and

$$P(X_1 > x_1, \ldots, X_m > x_m) \geq P(Y_1 > x_1, \ldots, Y_m > x_m), \tag{4.1.2}$$

$$P(X_1 \leq x_1, \ldots, X_m \leq x_m) \geq P(Y_1 \leq x_1, \ldots, Y_m \leq x_m), \tag{4.1.3}$$

for any $(x_1, \ldots, x_m)$.

It follows from Lemma 4.1.2 (4) that the supermodular dependence implies both upper and lower orthant dependence. Note that, if $\boldsymbol{X} \geq_{sm} \boldsymbol{Y}$, then $Cov(X_i, X_j) \geq Cov(Y_i, Y_j)$ for any $i \neq j$.

Lemma 4.1.3. Let $\boldsymbol{X} = (X_1, \ldots, X_m)$, and $\boldsymbol{Y} = (Y_1, \ldots, Y_m)$ be two random vectors.

1. If $\boldsymbol{X} \geq_{uod} \boldsymbol{Y}$ then $\min\{X_1, \ldots, X_m\} \geq_{st} \min\{Y_1, \ldots, Y_m\}$.
2. If $\boldsymbol{X} \geq_{lod} \boldsymbol{Y}$ then $\max\{X_1, \ldots, X_m\} \leq_{st} \max\{Y_1, \ldots, Y_m\}$.
3. If $\boldsymbol{X} \geq_{uod} \boldsymbol{Y}$ and $\boldsymbol{X} \geq_{lod} \boldsymbol{Y}$ then

$$E(\max\{X_1, \ldots, X_m\} - \min\{X_1, \ldots, X_m\}) \leq E(\max\{Y_1, \ldots, Y_m\} - \min\{Y_1, \ldots, Y_m\}).$$

Proof. Note that

$$P(\min\{X_1, \ldots, X_m\} > a) = P(X_1 > a, \ldots, X_m > a)$$

and

$$P(\min\{Y_1, \ldots, Y_m\} > a) = P(Y_1 > a, \ldots, Y_m > a).$$

Thus $\boldsymbol{X} \geq_{uod} \boldsymbol{Y}$ implies that

$$P(\min\{X_1, \ldots, X_m\} > a) \geq P(\min\{Y_1, \ldots, Y_m\} > a),$$

and (1) follows. (2) can be proved similarly, and (3) follows from (1) and (2). □

Example 4.1.4. Let Z be any random variable. The Lorenz inequality (Tchen 1980) implies that

$$(\underbrace{Z, \ldots, Z}_{m}) \geq_{sm} (Z_1, \ldots, Z_m)$$

where $Z_1, Z_2, \ldots, Z_m$ are i.i.d. random variables with the same distribution as that of Z. □

Example 4.1.5. Let $Z_1, \ldots, Z_s$ be independent random variables. Let $\mathbf{e}_s$ denote the vector of 1's with s dimension, $s \geq 1$. It follows from Example 4.1.4 and Lemma 4.1.2 (3) that

$$(Z_1\mathbf{e}_{m_1}, \ldots, Z_s\mathbf{e}_{m_s}) \geq_{sm} (\underbrace{Z_{11}, \ldots, Z_{1m_1}}_{m_1}, \ldots, \underbrace{Z_{s1}, \ldots, Z_{sm_s}}_{m_s}),$$

where $Z_{k1}, \ldots, Z_{km_k}$ are i.i.d. with the same distribution of Z_k, $k = 1, \ldots, s$. □

Since the supermodular dependence is stronger than upper and lower orthant dependence, Examples 4.1.4 and 4.1.5 also provide examples for the upper and lower orthant dependence. The following example, due to Šidák (1968), presents a random vector with the upper orthant dependence.

Example 4.1.6. Let $\boldsymbol{X} = (X_1, \ldots, X_m)$ have a normal distribution with a mean vector of μ and a covariance matrix of $\Sigma = (\sigma_{ij})$. Then

$$(|X_1|, \ldots, |X_m|) \geq_{uod} (Y_1, \ldots, Y_m)$$

where $Y_1, \ldots, Y_m$ are independent having distributions of $|X_1|, \ldots, |X_m|$ respectively. □

4.2 Dependence Comparisons of Multivariate Process Models

Consider two multivariate compound point process models $\mathcal{M}_1$ and $\mathcal{M}_2$ introduced in Chapter 1. To compare the effect of the dependence of damage sizes on the ruin probabilities, we suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same claim event arrival process $\{N(t), t \geq 0\}$, same drift functions $\lambda_j(n)$, $1 \leq j \leq m$, and same initial reserves u_j, $1 \leq j \leq m$, but different damage size vectors $\boldsymbol{X}_n = (X_{n,1}, \ldots, X_{n,m})$ and $\boldsymbol{Y}_n = (Y_{n,1}, \ldots, Y_{n,m})$, respectively. Let $\psi^X_{and}(u_1, \ldots, u_m)$ $(\psi^Y_{and}(u_1, \ldots, u_m))$, $\psi^X_{or}(u_1, \ldots, u_m)$ $(\psi^Y_{or}(u_1, \ldots, u_m))$, and $\psi^X_{sim}(u_1, \ldots, u_m)$ $(\psi^Y_{sim}(u_1, \ldots, u_m))$ denote the ruin probabilities of types (2.1.6), (2.1.7), and (2.1.8), respectively, in model $\mathcal{M}_1$ $(\mathcal{M}_2)$.

Theorem 4.2.1. If $\boldsymbol{X}_n \leq_{sm} \boldsymbol{Y}_n$, then we have, for any nonnegative $u_1, \ldots, u_m$,

1. $\psi^X_{and}(u_1, \ldots, u_m) \leq \psi^Y_{and}(u_1, \ldots, u_m)$,

2. $\psi^X_{or}(u_1, \ldots, u_m) \geq \psi^Y_{or}(u_1, \ldots, u_m)$, and

3. $\psi^X_{sim}(u_1, \ldots, u_m) \leq \psi^Y_{sim}(u_1, \ldots, u_m)$.

Proof. (1) It suffices to show that given that $N(t) = n(t), t \geq 0$,

$$\psi^X_{and}(u_1, \ldots, u_m) \leq \psi^Y_{and}(u_1, \ldots, u_m). \tag{4.2.1}$$

Without loss of generality, we assume that $\{\boldsymbol{X}_n, n \geq 1\}$ and $\{\boldsymbol{Y}_n, n \geq 1\}$ are independent. For fixed positive integer k, let

$$\begin{aligned} \boldsymbol{Z}_n &= \boldsymbol{Y}_n, n = 1, \ldots, k \\ \boldsymbol{Z}_n &= \boldsymbol{X}_n, n > k. \end{aligned}$$

Let $\psi^Z_k(u_1, \ldots, u_m)$ denote the ruin probabilities of type (2.1.6) in the multivariate compound point model with the shock arrival process $N(t)$, drift rates $\lambda_j(N(t))$, $1 \leq j \leq m$, initial reserves u_j, $1 \leq j \leq m$, and damage size vectors $\{\boldsymbol{Z}_n, n \geq 1\}$. Also let

$$\begin{aligned} S^X_j(t) &= \sum_{n=1}^{N(t)} X_{n,j} - \int_0^t \lambda_j(N(s))ds, \; j = 1, \ldots, m \\ S^Y_j(t) &= \sum_{n=1}^{N(t)} Y_{n,j} - \int_0^t \lambda_j(N(s))ds, \; j = 1, \ldots, m \\ S_j(t) &= \sum_{n=1}^{N(t)} Z_{n,j} - \int_0^t \lambda_j(N(s))ds, \; j = 1, \ldots, m. \end{aligned}$$

Conditioning on $\boldsymbol{X}_n = \boldsymbol{x}_n$, $n > k$, $\sup_{0 \leq t < \infty} S^X_j(t)$ is an increasing function of $X_{1,j}, \ldots, X_{k,j}$, and $\sup_{0 \leq t < \infty} S_j(t)$ is an increasing function of $Y_{1,j}, \ldots, Y_{k,j}$, $1 \leq j \leq m$. Since $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ are i.i.d., and $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k$ are i.i.d, and $\boldsymbol{X}_n \leq_{sm} \boldsymbol{Y}_n$, we invoke Lemma 4.1.2 (3) k times, and obtain that conditioning on $\boldsymbol{X}_n = \boldsymbol{x}_n$, $n > k$,

$$\left(\sup_{0 \leq t < \infty} S^X_1(t), \ldots, \sup_{0 \leq t < \infty} S^X_m(t) \right) \leq_{sm} \left(\sup_{0 \leq t < \infty} S_1(t), \ldots, \sup_{0 \leq t < \infty} S_m(t) \right).$$

It follows from unconditioning and (4.1.2) that for any k

$$\psi^X_{and}(u_1, \ldots, u_m) \leq \psi^Z_k(u_1, \ldots, u_m).$$

Observe that as $k \to \infty$, $\psi_k^Z(u_1, \ldots, u_m)$ converges to $\psi_{and}^Y(u_1, \ldots, u_m)$ for any $u_1, \ldots, u_m$. Thus, we establish (4.2.1) conditioning on $N(t) = n(t), t \geq 0$.

(2) Note that for any two events A and B, we have $P(A \text{ or } B) = 1 - P((A \text{ or } B)')$ = 1 - $P(A' \text{ and } B')$. Using a similar idea as in (1) above (using (4.1.3), instead of (4.1.2)), we can also show that

$$P\left(\sup_{0\leq t<\infty} S_1^X(t) \leq u_1, \ldots, \sup_{0\leq t<\infty} S_m^X(t) \leq u_1\right) \leq P\left(\sup_{0\leq t<\infty} S_1^Y(t) \leq u_1, \ldots, \sup_{0\leq t<\infty} S_m^Y(t) \leq u_1\right).$$

Therefore,

$$\begin{aligned}
\psi_{or}^X(u_1, \ldots, u_m) &= 1 - P\left(\sup_{0\leq t<\infty} S_1^X(t) \leq u_1, \ldots, \sup_{0\leq t<\infty} S_m^X(t) \leq u_1\right) \\
&\geq 1 - P\left(\sup_{0\leq t<\infty} S_1^Y(t) \leq u_1, \ldots, \sup_{0\leq t<\infty} S_m^Y(t) \leq u_1\right) = \psi_{or}^Y(u_1, \ldots, u_m).
\end{aligned}$$

(3) Notice that $\psi_{sim}(u_1, \ldots, u_m)$ is the probability that ruin occurs at all the components at the *same time*, and unlike (2.1.6) and (2.1.7), is not a separate functional of the damage processes of these components. Thus, in this case, we need some extra work.

Let

$$\bar{S}_j^X(t) = S_j^X(t) - u_j,\ \bar{S}_j^Y(t) = S_j^Y(t) - u_j,\ 1 \leq j \leq m.$$

Also let

$$\begin{aligned}
\bar{S}_{(1)}^X(t) &= \min\left\{\bar{S}_1^X(t), \ldots, \bar{S}_m^X(t)\right\}, \\
\bar{S}_{(1)}^Y(t) &= \min\left\{\bar{S}_1^Y(t), \ldots, \bar{S}_m^Y(t)\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\psi_{sim}^X(u_1, \ldots, u_m) &= 1 - P\left(\sup_{0\leq t<\infty} \bar{S}_{(1)}^X(t) \leq 0\right) = 1 - P\left(\bar{S}_{(1)}^X(t) \leq 0 \text{ for all } t \geq 0\right), \\
\psi_{sim}^Y(u_1, \ldots, u_m) &= 1 - P\left(\sup_{0\leq t<\infty} \bar{S}_{(1)}^Y(t) \leq 0\right) = 1 - P\left(\bar{S}_{(1)}^Y(t) \leq 0 \text{ for all } t \geq 0\right),
\end{aligned}$$

we need to show that

$$P\left(\bar{S}_{(1)}^X(t) \leq 0 \text{ for all } t \geq 0\right) \geq P\left(\bar{S}_{(1)}^Y(t) \leq 0 \text{ for all } t \geq 0\right).$$

Since the sample paths of the counting process $\{N(t), t \geq 0\}$ are right-continuous with left-limits, it suffices to show that for any $0 \le t_1 \le t_2 \le \cdots \le t_l < \infty$,

$$P\left(\bar{S}^X_{(1)}(t_1) \le 0, \ldots, \bar{S}^X_{(1)}(t_l) \le 0\right) \ge P\left(\bar{S}^Y_{(1)}(t_1) \le 0, \ldots, \bar{S}^Y_{(1)}(t_l) \le 0\right),$$

which can be rephrased as

$$P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\left\{\bar{S}^X_j(t_i) \le 0\right\}\right) \ge P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\left\{\bar{S}^Y_j(t_i) \le 0\right\}\right). \tag{4.2.2}$$

We first observe that for any real numbers $a_1, \ldots, a_l$ and any n, we have,

$$\begin{aligned}
P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{X_{n,j} \le a_i\}\right) &= P\left(\bigcap_{i=1}^{l}\{\min\{X_{n,1}, \ldots, X_{n,m}\} \le a_i\}\right) \\
&= P\left(\min\{X_{n,1}, \ldots, X_{n,m}\} \le \min\{a_1, \ldots, a_l\}\right) \\
&= 1 - P\left(X_{n,1} > \min\{a_1, \ldots, a_l\}, \ldots, X_{n,m} > \min\{a_1, \ldots, a_l\}\right) \\
&\ge 1 - P\left(Y_{n,1} > \min\{a_1, \ldots, a_l\}, \ldots, Y_{n,m} > \min\{a_1, \ldots, a_l\}\right) \\
&= P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{Y_{n,j} \le a_i\}\right),
\end{aligned}$$

where the inequality follows from (4.1.2). Thus, for any strictly increasing functions $g_1, \ldots, g_l$ and any n, we have

$$\begin{aligned}
P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{g_i(X_{n,j}) \le 0\}\right) &= P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\left\{X_{n,j} \le g_i^{-1}(0)\right\}\right) \\
&\ge P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\left\{Y_{n,j} \le g_i^{-1}(0)\right\}\right) \\
&= P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{g_i(Y_{n,j}) \le 0\}\right).
\end{aligned} \tag{4.2.3}$$

Conditioning on $N(t) = n(t), t \ge 0$, $\bar{S}^X_j(t_1), \ldots, \bar{S}^X_j(t_l)$ are strictly increasing functions of $X_{n,j}$, $1 \le n \le k$, for certain k, where k is finite due to the fact that $\{N(t), t \ge 0\}$ is non-explosive. Similarly, $\bar{S}^Y_j(t_1), \ldots, \bar{S}^Y_j(t_l)$ are strictly increasing functions of $Y_{n,j}$, $1 \le n \le k$. Since $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ are i.i.d., and $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k$ are i.i.d, and $\boldsymbol{X}_n \le_{sm} \boldsymbol{Y}_n$, we invoke (4.2.3) k times, and obtain (4.2.2) conditioning on $N(t) = n(t), t \ge 0$. Finally, unconditioning yields (4.2.2). □

Note that $\psi^X_{sim}(u_1, \ldots, u_m) \leq \psi^Y_{and}(u_1, \ldots, u_m) \leq \psi^X_{or}(u_1, \ldots, u_m)$ for any $u_1, \ldots, u_m$. Theorem 4.2.1 shows that, as the damage size vector becomes more correlated in the sense of supermodular order, both $\psi^X_{sim}(u_1, \ldots, u_m)$ and $\psi^Y_{and}(u_1, \ldots, u_m)$ increase, and $\psi^X_{or}(u_1, \ldots, u_m)$ decreases.

Example 4.2.2. Consider a multivariate compound Poisson model with constant drift functions $\lambda_j = p$, $1 \leq j \leq m$, Poisson shock arrival process $N = \{N(t), t \geq 0\}$ with rate λ, and damage size vector $(X_{n,1}, \ldots, X_{n,m})$. Assume that $(X_{n,1}, \ldots, X_{n,m})$ has a joint distribution such that all the one dimensional marginals have an exponential distribution with rate β. Even when we know the dependence structure of $(X_{n,1}, \ldots, X_{n,m})$, the ruin probabilities $\psi^X_{and}(u_1, \ldots, u_m)$, $\psi^X_{or}(u_1, \ldots, u_m)$ and $\psi^X_{sim}(u_1, \ldots, u_m)$ from (2.1.6)-(2.1.8) still have no closed formulas. To find computable bounds, we utilize Theorem 4.2.1 and Example 4.1.4. First observe from Example 4.1.4 that

$$(X_{n,1}, \ldots, X_{n,m}) \leq_{sm} (\underbrace{X_{n,1}, \ldots, X_{n,1}}_{m}).$$

It follows from Theorem 4.2.1 that

$$\psi^X_{and}(u_1, \ldots, u_m) \leq \psi_{and}(u_1, \ldots, u_m),$$

$$\psi^X_{or}(u_1, \ldots, u_m) \geq \psi_{or}(u_1, \ldots, u_m),$$

$$\psi^X_{sim}(u_1, \ldots, u_m) \leq \psi_{sim}(u_1, \ldots, u_m),$$

where $\psi_{and}(u_1, \ldots, u_m)$, $\psi_{or}(u_1, \ldots, u_m)$ and $\psi_{sim}(u_1, \ldots, u_m)$ are the ruin probabilities of type (2.1.6), (2.1.7) and (2.1.8), respectively, with damage size vector $(X_{n,1}, \ldots, X_{n,1})$. Obviously, we have

$$\psi_{and}(u_1, \ldots, u_m) = \psi(\max\{u_1, \ldots, u_m\}),$$

$$\psi_{or}(u_1, \ldots, u_m) = \psi(\min\{u_1, \ldots, u_m\}),$$

$$\psi_{sim}(u_1, \ldots, u_m) = \psi(\max\{u_1, \ldots, u_m\}),$$

where $\psi(u)$ is the univariate ruin probability (2.1.9) with Poisson shock arrival process, and exponential distribution damage size. Note that $\psi_{sim} = \psi_{and} \leq \psi_{or}$. This ruin probability can be calculated explicitly (2.2.1). Thus we have

$$\psi^X_{and}(u_1, \ldots, u_m) \leq \frac{1}{1+\theta} \exp\left(-\frac{\theta\beta}{1+\theta} \max\{u_1, \ldots, u_m\}\right),$$

$$\psi^X_{or}(u_1,\dots,u_m) \ge \frac{1}{1+\theta}\exp\left(-\frac{\theta\beta}{1+\theta}\min\{u_1,\dots,u_m\}\right),$$

$$\psi^X_{sim}(u_1,\dots,u_m) \le \frac{1}{1+\theta}\exp\left(-\frac{\theta\beta}{1+\theta}\max\{u_1,\dots,u_m\}\right),$$

where $\theta = p\beta/\lambda - 1 > 0$ is known as the relative security loading parameter. □

For the weaker dependence comparisons $\le_{uod}$ and $\le_{lod}$, the weaker results can be similarly established.

Theorem 4.2.3. If $\boldsymbol{X}_n \le_{uod} \boldsymbol{Y}_n$, then we have, for any nonnegative $u_1,\dots,u_m$,

1. $\psi^X_{and}(u_1,\dots,u_m) \le \psi^Y_{and}(u_1,\dots,u_m)$, and
2. $\psi^X_{sim}(u_1,\dots,u_m) \le \psi^Y_{sim}(u_1,\dots,u_m)$.

Proof. (1) It suffices to show that given that $N(t) = n(t), t \ge 0$,

$$\psi^X_{and}(u_1,\dots,u_m) \le \psi^Y_{and}(u_1,\dots,u_m). \tag{4.2.4}$$

Without loss of generality, we assume that $\{\boldsymbol{X}_n, n \ge 1\}$ and $\{\boldsymbol{Y}_n, n \ge 1\}$ are independent. For fixed positive integer k, let

$$\begin{aligned} \boldsymbol{Z}_n &= \boldsymbol{Y}_n, n = 1,\dots,k \\ \boldsymbol{Z}_n &= \boldsymbol{X}_n, n > k. \end{aligned}$$

Let $\psi^Z_k(u_1,\dots,u_m)$ denote the ruin probabilities of type (2.1.6) in the multivariate compound point model with the shock arrival process $N(t)$, drift rates $\lambda_j(N(t))$, $1 \le j \le m$, initial reserves u_j, $1 \le j \le m$, and damage size vectors $\{\boldsymbol{Z}_n, n \ge 1\}$. Also let

$$\begin{aligned} S^X_j(t) &= \sum_{n=1}^{N(t)} X_{n,j} - \int_0^t \lambda_j(N(s))ds,\ j = 1,\dots,m \\ S^Y_j(t) &= \sum_{n=1}^{N(t)} Y_{n,j} - \int_0^t \lambda_j(N(s))ds,\ j = 1,\dots,m \\ S_j(t) &= \sum_{n=1}^{N(t)} Z_{n,j} - \int_0^t \lambda_j(N(s))ds,\ j = 1,\dots,m. \end{aligned}$$

Conditioning on $\boldsymbol{X}_n = \boldsymbol{x}_n$, $n > k$, $\sup_{0\le t<\infty} S^X_j(t)$ is an increasing function of $X_{1,j},\dots,X_{k,j}$, and $\sup_{0\le t<\infty} S_j(t)$ is an increasing function of $Y_{1,j},\dots,Y_{k,j}$, $1 \le j \le m$. Since $\boldsymbol{X}_1,\dots,\boldsymbol{X}_k$

are i.i.d., and $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k$ are i.i.d, and $\boldsymbol{X}_n \le_{sm} \boldsymbol{Y}_n$, we invoke Lemma 4.1.2 (1) k times, and obtain that conditioning on $\boldsymbol{X}_n = \boldsymbol{x}_n$, $n > k$,

$$\left(\sup_{0 \le t < \infty} S_1^X(t), \ldots, \sup_{0 \le t < \infty} S_m^X(t) \right) \le_{uod} \left(\sup_{0 \le t < \infty} S_1(t), \ldots, \sup_{0 \le t < \infty} S_m(t) \right).$$

It follows from unconditioning and (4.1.2) that for any k

$$\psi_{and}^X(u_1, \ldots, u_m) \le \psi_k^Z(u_1, \ldots, u_m).$$

Observe that as $k \to \infty$, $\psi_k^Z(u_1, \ldots, u_m)$ converges to $\psi_{and}^Y(u_1, \ldots, u_m)$ for any $u_1, \ldots, u_m$. Thus, we establish (4.2.1) conditioning on $N(t) = n(t), t \ge 0$.

(2) Let

$$\bar{S}_j^X(t) = S_j^X(t) - u_j,\ \bar{S}_j^Y(t) = S_j^Y(t) - u_j,\ 1 \le j \le m.$$

Also let

$$\bar{S}_{(1)}^X(t) = \min\left\{\bar{S}_1^X(t), \ldots, \bar{S}_m^X(t)\right\},$$
$$\bar{S}_{(1)}^Y(t) = \min\left\{\bar{S}_1^Y(t), \ldots, \bar{S}_m^Y(t)\right\}.$$

Since

$$\begin{aligned} \psi_{sim}^X(u_1, \ldots, u_m) &= 1 - P\left(\sup_{0 \le t < \infty} \bar{S}_{(1)}^X(t) \le 0 \right) = 1 - P\left(\bar{S}_{(1)}^X(t) \le 0 \text{ for all } t \ge 0 \right), \\ \psi_{sim}^Y(u_1, \ldots, u_m) &= 1 - P\left(\sup_{0 \le t < \infty} \bar{S}_{(1)}^Y(t) \le 0 \right) = 1 - P\left(\bar{S}_{(1)}^Y(t) \le 0 \text{ for all } t \ge 0 \right), \end{aligned}$$

we need to show that

$$P\left(\bar{S}_{(1)}^X(t) \le 0 \text{ for all } t \ge 0 \right) \ge P\left(\bar{S}_{(1)}^Y(t) \le 0 \text{ for all } t \ge 0 \right).$$

Since the sample paths of the counting process $\{N(t), t \ge 0\}$ are right-continuous with left-limits, it suffices to show that for any $0 \le t_1 \le t_2 \le \cdots \le t_l < \infty$,

$$P\left(\bar{S}_{(1)}^X(t_1) \le 0, \ldots, \bar{S}_{(1)}^X(t_l) \le 0 \right) \ge P\left(\bar{S}_{(1)}^Y(t_1) \le 0, \ldots, \bar{S}_{(1)}^Y(t_l) \le 0 \right),$$

which can be rephrased as

$$P\left(\bigcap_{i=1}^{l} \bigcup_{j=1}^{m} \left\{ \bar{S}_j^X(t_i) \le 0 \right\} \right) \ge P\left(\bigcap_{i=1}^{l} \bigcup_{j=1}^{m} \left\{ \bar{S}_j^Y(t_i) \le 0 \right\} \right). \tag{4.2.5}$$

We first observe that for any real numbers $a_1, \ldots, a_l$ and any n, we have,

$$\begin{aligned} P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{X_{n,j} \le a_i\}\right) &= P\left(\bigcap_{i=1}^{l}\{\min\{X_{n,1},\ldots,X_{n,m}\} \le a_i\}\right) \\ &= P\left(\min\{X_{n,1},\ldots,X_{n,m}\} \le \min\{a_1,\ldots,a_l\}\right) \\ &= 1 - P\left(X_{n,1} > \min\{a_1,\ldots,a_l\},\ldots,X_{n,m} > \min\{a_1,\ldots,a_l\}\right) \\ &\ge 1 - P\left(Y_{n,1} > \min\{a_1,\ldots,a_l\},\ldots,Y_{n,m} > \min\{a_1,\ldots,a_l\}\right) \\ &= P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{Y_{n,j} \le a_i\}\right), \end{aligned}$$

where the inequality follows from (4.1.2). Thus, for any strictly increasing functions $g_1, \ldots, g_l$ and any n, we have

$$\begin{aligned} P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{g_i(X_{n,j}) \le 0\}\right) &= P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\left\{X_{n,j} \le g_i^{-1}(0)\right\}\right) \\ &\ge P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\left\{Y_{n,j} \le g_i^{-1}(0)\right\}\right) \\ &= P\left(\bigcap_{i=1}^{l}\bigcup_{j=1}^{m}\{g_i(Y_{n,j}) \le 0\}\right). \end{aligned} \tag{4.2.6}$$

Conditioning on $N(t) = n(t), t \ge 0$, $\bar{S}_j^X(t_1), \ldots, \bar{S}_j^X(t_l)$ are strictly increasing functions of $X_{n,j}$, $1 \le n \le k$, for certain k, where k is finite due to the fact that $\{N(t), t \ge 0\}$ is non-explosive. Similarly, $\bar{S}_j^Y(t_1), \ldots, \bar{S}_j^Y(t_l)$ are strictly increasing functions of $Y_{n,j}$, $1 \le n \le k$. Since $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ are i.i.d., and $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k$ are i.i.d, and $\boldsymbol{X}_n \le_{sm} \boldsymbol{Y}_n$, we invoke (4.2.6) k times, and obtain (4.2.5) conditioning on $N(t) = n(t), t \ge 0$. Finally, unconditioning yields (4.2.5). □

Theorem 4.2.4. If $\boldsymbol{X}_n \le_{lod} \boldsymbol{Y}_n$, then we have, for any nonnegative $u_1, \ldots, u_m$,

$$\psi_{or}^X(u_1, \ldots, u_m) \ge \psi_{or}^Y(u_1, \ldots, u_m).$$

Proof. It suffices to show that given that $N(t) = n(t), t \ge 0$,

$$\psi_{or}^X(u_1, \ldots, u_m) \ge \psi_{or}^Y(u_1, \ldots, u_m). \tag{4.2.7}$$

Without loss of generality, we assume that $\{\boldsymbol{X}_n, n \ge 1\}$ and $\{\boldsymbol{Y}_n, n \ge 1\}$ are independent.

For fixed positive integer k, let

$$\begin{aligned} \boldsymbol{Z}_n &= \boldsymbol{Y}_n, n = 1, \dots, k \\ \boldsymbol{Z}_n &= \boldsymbol{X}_n, n > k. \end{aligned}$$

Let $\psi_k^Z(u_1, \dots, u_m)$ denote the ruin probabilities of type (2.1.7) in the multivariate compound point model with the shock arrival process $N(t)$, drift rates $\lambda_j(N(t))$, $1 \le j \le m$, initial reserves u_j, $1 \le j \le m$, and damage size vectors $\{\boldsymbol{Z}_n, n \ge 1\}$. Also let

$$\begin{aligned} S_j^X(t) &= \sum_{n=1}^{N(t)} X_{n,j} - \int_0^t \lambda_j(N(s))ds,\ j = 1, \dots, m \\ S_j^Y(t) &= \sum_{n=1}^{N(t)} Y_{n,j} - \int_0^t \lambda_j(N(s))ds,\ j = 1, \dots, m \\ S_j(t) &= \sum_{n=1}^{N(t)} Z_{n,j} - \int_0^t \lambda_j(N(s))ds,\ j = 1, \dots, m. \end{aligned}$$

Conditioning on $\boldsymbol{X}_n = \boldsymbol{x}_n$, $n > k$, $\sup_{0 \le t < \infty} S_j^X(t)$ is an increasing function of $X_{1,j}, \dots, X_{k,j}$, and $\sup_{0 \le t < \infty} S_j(t)$ is an increasing function of $Y_{1,j}, \dots, Y_{k,j}$, $1 \le j \le m$. Since $\boldsymbol{X}_1, \dots, \boldsymbol{X}_k$ are i.i.d., and $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_k$ are i.i.d, and $\boldsymbol{X}_n \le_{lod} \boldsymbol{Y}_n$, we invoke Lemma 4.1.2 (2) k times, and obtain that conditioning on $\boldsymbol{X}_n = \boldsymbol{x}_n$, $n > k$,

$$\left(\sup_{0 \le t < \infty} S_1^X(t), \dots, \sup_{0 \le t < \infty} S_m^X(t) \right) \le_{lod} \left(\sup_{0 \le t < \infty} S_1(t), \dots, \sup_{0 \le t < \infty} S_m(t) \right).$$

It follows from unconditioning and (4.1.3) that for any k

$$\begin{aligned} & P\left(\sup_{0 \le t < \infty} S_1^X(t) \le u_1, \dots, \sup_{0 \le t < \infty} S_m^X(t) \le u_1\right) \\ \le\ & P\left(\sup_{0 \le t < \infty} S_1^Y(t) \le u_1, \dots, \sup_{0 \le t < \infty} S_m^Y(t) \le u_1\right). \end{aligned} \tag{4.2.8}$$

Let $k \to \infty$, (4.2.8) still holds. Therefore,

$$\begin{aligned} \psi_{or}^X(u_1, \dots, u_m) &= 1 - P\left(\sup_{0 \le t < \infty} S_1^X(t) \le u_1, \dots, \sup_{0 \le t < \infty} S_m^X(t) \le u_1 \right) \\ &\ge 1 - P\left(\sup_{0 \le t < \infty} S_1^Y(t) \le u_1, \dots, \sup_{0 \le t < \infty} S_m^Y(t) \le u_1 \right) \\ &= \psi_{or}^Y(u_1, \dots, u_m), \end{aligned}$$

conditioning on $N(t) = n(t), t \geq 0$. Finally, unconditioning yields the inequality. $\square$

Chapter 5

Stochastic Bounds and Simulations

Since the ruin probabilities have no closed formulas, we seek their computable bounds in this chapter. Two approaches are used. First, we derive the bounds using univariate compound point processes, and then approach utilizes the positive dependence concepts. All the bounds discussed in this chapter depend on the dependence structure of damage vectors, and thus the effect of dependence on these bounds are also discussed.

To facilitate the computations of these bounds, we utilize the multivariate phase type distributions. The multivariate phase type distributions include many well-known distributions, such as Marshall-Olkin distributions, and can be also used to approximate any multivariate life distributions. The explicit computations of the bounds for bivariate and trivariate Marshall-Olkin distributions are obtained, and extensive simulation results are also presented to illustrate the results.

5.1 Upper and Lower Bounds Using Univariate Ruin Probabilities

Our bounding strategy is to bound the multivariate ruin probabilities (2.1.5)-(2.1.8) by some univariate ruin probabilities, which can be calculated under certain conditions for the phase type distributed damages.

Consider a multivariate compound point model (2.1.2). Let the drift function of the j-th component $\lambda_j(n)$ be bounded, that is, $0 \leq p_j \leq \lambda_j(n) \leq P_j < \infty$ for all $n \geq 1$, where

p_j and P_j are constants, $1 \le j \le m$. Let

$$\begin{aligned} X_{(1),n} &= \min\{X_{n,1}, \ldots, X_{n,m}\}, & X_{(m),n} &= \max\{X_{n,1}, \ldots, X_{n,m}\}, \\ p_{(1)} &= \min\{p_1, \ldots, p_m\}, & p_{(m)} &= \max\{P_1, \ldots, P_m\}, \\ u_{(1)} &= \min\{u_1, \ldots, u_m\}, & u_{(m)} &= \max\{u_1, \ldots, u_m\}. \end{aligned}$$

Also let

$$\psi_{min}(u) = P\left(\sup_{0 \le t < \infty}\left(\sum_{n=1}^{N(t)} X_{(1),n} - p_{(m)}t\right) > u\right), \tag{5.1.1}$$

$$\psi_{max}(u) = P\left(\sup_{0 \le t < \infty}\left(\sum_{n=1}^{N(t)} X_{(m),n} - p_{(1)}t\right) > u\right). \tag{5.1.2}$$

Clearly, for any nonnegative $(u_1, \ldots, u_m)$,

$$\psi_{min}(u_{(m)}) \le \psi_{sim}(u_1, \ldots, u_m) \le \psi_{or}(u_1, \ldots, u_m) \le \psi_{max}(u_{(1)}).$$

Consider now the following two ruin probabilities, for any

$$\boldsymbol{a} \in \mathcal{A} = \left\{(a_1, \ldots, a_m) : a_j \ge 0, 1 \le j \le m, \text{ and } \sum_{j=1}^{m} a_j > 0\right\},$$

let

$$\psi_{\boldsymbol{a}}^{p}(u) = P\left(\sup_{0 \le t < \infty}\left\{\sum_{n=1}^{N(t)}\left(\sum_{j=1}^{m} a_j X_{n,j}\right) - \left(\sum_{j=1}^{m} a_j p_j\right)t\right\} > u\right), \tag{5.1.3}$$

$$\psi_{\boldsymbol{a}}^{P}(u) = P\left(\sup_{0 \le t < \infty}\left\{\sum_{n=1}^{N(t)}\left(\sum_{j=1}^{m} a_j X_{n,j}\right) - \left(\sum_{j=1}^{m} a_j P_j\right)t\right\} > u\right), \tag{5.1.4}$$

$$\psi_{sum}^{p}(u) = P\left(\sup_{0 \le t < \infty}\left\{\sum_{n=1}^{N(t)}\left(\sum_{j=1}^{m} X_{n,j}\right) - \left(\sum_{j=1}^{m} p_j\right)t\right\} > u\right), \tag{5.1.5}$$

$$\psi_{sum}^{P}(u) = P\left(\sup_{0 \le t < \infty}\left\{\sum_{n=1}^{N(t)}\left(\sum_{j=1}^{m} X_{n,j}\right) - \left(\sum_{j=1}^{m} P_j\right)t\right\} > u\right). \tag{5.1.6}$$

Using the notations in (2.1.2), we observe that

$$\psi^p_{\boldsymbol{a}}\left(\sum_{j=1}^m a_j u_j\right) = P\left(\sup_{0\le t<\infty}\left\{\sum_{j=1}^m a_j\left(\sum_{n=1}^{N(t)} X_{n,j} - p_j t - u_j\right)\right\} > 0\right), \qquad (5.1.7)$$

$$\psi^P_{\boldsymbol{a}}\left(\sum_{j=1}^m a_j u_j\right) = P\left(\sup_{0\le t<\infty}\left\{\sum_{j=1}^m a_j\left(\sum_{n=1}^{N(t)} X_{n,j} - P_j t - u_j\right)\right\} > 0\right). \qquad (5.1.8)$$

We point that for any $\boldsymbol{a} \in \mathcal{A}$ if $p_j > 0$, $\mu_j \ge 0$, and $t\,p_j > E(N(t))\,E(X_{j,1}) > 0$ for all $j = 1, ..., m$ and $t > 0$, then $\sum_{j=1}^m a_j p_j > 0$, $\sum_{j=1}^m a_j u_j \ge 0$, and $t\sum_{j=1}^m a_j p_j > E(N(t))\sum_{j=1}^m a_j\,E(X_{j,1}) > 0$. Hence, there are positive drift rates, nonnegative initial capitals, and positive relative security loadings in the ruin probabilities (5.1.3)-(5.1.8).

On one hand, for any $(a_1, \ldots, a_m) \in \mathcal{A}$ and some $t > 0$, the event $\{S_1(t) > u_1, ..., S_m(t) > u_m\}$ implies the event $\left\{\sum_{j=1}^m a_j(S_j(t) - u_j) > 0\right\}$ holds. Hence,

$$\psi_{sim}(u_1, \ldots, u_m) \le \psi^p_{\boldsymbol{a}}\left(\sum_{j=1}^m a_j u_j\right) \qquad (5.1.9)$$

for any $(a_1, \ldots, a_m) \in \mathcal{A}$.

On the other hand, for any $(a_1, \ldots, a_m) \in \mathcal{A}$ and some $t > 0$, the event $\{\sum_{j=1}^m a_j(S_j(t) - u_j) > 0\}$ implies that the event $\{S_j(t) - u_j > 0\}$ for at least one j holds. Thus, we also have

$$\psi^P_{\boldsymbol{a}}\left(\sum_{j=1}^m a_j u_j\right) \le \psi_{or}(u_1, \ldots, u_m) \qquad (5.1.10)$$

for any $(a_1, \ldots, a_m) \in \mathcal{A}$. We summarize all these results in the following proposition.

Proposition 5.1.1. Let $\mathcal{A} = \left\{(a_1, \ldots, a_m) : a_j \ge 0,\ 1 \le j \le m, \text{ and } \sum_{j=1}^m a_j > 0\right\}$.

1. $\psi_{min}(u_{(m)}) \le \psi_{sim}(u_1, \ldots, u_m) \le \inf_{\boldsymbol{a}\in\mathcal{A}} \psi^p_{\boldsymbol{a}}\left(\sum_{j=1}^m a_j u_j\right)$.

2. $\sup_{\boldsymbol{a}\in\mathcal{A}} \psi^P_{\boldsymbol{a}}\left(\sum_{j=1}^m a_j u_j\right) \le \psi_{or}(u_1, \ldots, u_m) \le \psi_{max}(u_{(1)})$.

3. In particular,

$$\psi_{min}(u_{(m)}) \le \psi_{sim}(u_1, \ldots, u_m) \le \psi^p_{sum}\left(\sum_{j=1}^m u_j\right).$$

$$\psi^P_{sum}\left(\sum_{j=1}^m u_j\right) \le \psi_{or}(u_1, \dots, u_m) \le \psi_{max}(u_{(1)}).$$

The upper bound in (1) and the lower bound in (2) of Proposition 5.1.1 have been discussed in Chan et al. (2003) for the models with constant drift functions. The bounds presented in Proposition 5.1.1 are the ruin probabilities of univariate risk processes, but depend on the dependence structure of the underlying multivariate compound point process. To see this, consider two multivariate compound point process models $\mathcal{M}_1$ and $\mathcal{M}_2$ introduced in Chapter 2. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same shock arrival process $\{N(t), t \ge 0\}$, same drift rate functions $\lambda_j(n)$, $1 \le j \le m$, and same initial reserves u_j, $1 \le j \le m$, but different damage size vectors $\boldsymbol{X}_n = (X_{n,1}, \dots, X_{n,m})$ and $\boldsymbol{Y}_n = (Y_{n,1}, \dots, Y_{n,m})$, respectively. Let $\psi^X_{min}(u)$ $(\psi^Y_{min}(u))$, $\psi^X_{max}(u)$ $(\psi^Y_{max}(u))$, and $\psi^{pX}_{sum}(u)$ and $\psi^{PX}_{sum}(u)$ $(\psi^{pY}_{sum}(u)$ and $\psi^{PY}_{sum}(u))$ denote the ruin probabilities of types (5.1.1), (5.1.2), and (5.1.5) and (5.1.6), respectively, in model $\mathcal{M}_1$ $(\mathcal{M}_2)$.

Proposition 5.1.2. If $\boldsymbol{X}_n \ge_{uod} \boldsymbol{Y}_n$, then we have, for any nonnegative u, $\psi^X_{min}(u) \ge \psi^Y_{min}(u)$.

Proof. Clearly, $\boldsymbol{X}_n \ge_{uod} \boldsymbol{Y}_n$ implies that

$$X_{(1),n} \ge_{st} Y_{(1),n}.$$

Thus, result follows from the fact that $\psi^X_{min}(u)$ is the increasing function of $X_{(1),n}$, $n \ge 1$. □

Proposition 5.1.3. If $\boldsymbol{X}_n \ge_{lod} \boldsymbol{Y}_n$, then we have, for any nonnegative u, $\psi^X_{max}(u) \le \psi^Y_{max}(u)$.

Proof. Clearly, $\boldsymbol{X}_n \ge_{lod} \boldsymbol{Y}_n$ implies that

$$X_{(m),n} \le_{st} Y_{(m),n}.$$

Thus, result follows from the fact that $\psi^X_{max}(u)$ is the increasing function of $X_{(m),n}$, $n \ge 1$. □

Proposition 5.1.4. If $\boldsymbol{X}_n \ge_{sm} \boldsymbol{Y}_n$, then we have, for any nonnegative u,

1. $\psi^X_{min}(u) \ge \psi^Y_{min}(u)$,

2. $\psi^X_{max}(u) \le \psi^Y_{max}(u)$, and

3. $\psi^{pX}_{sum}(u) \ge \psi^{pY}_{sum}(u)$ and $\psi^{PX}_{sum}(u) \ge \psi^{PY}_{sum}(u)$.

Proof. Clearly, $\boldsymbol{X}_n \ge_{sm} \boldsymbol{Y}_n$ implies that

$$X_{(1),n} \ge_{st} Y_{(1),n},\ X_{(m),n} \le_{st} Y_{(m),n}.$$

Thus, (1) and (2) follow from the fact that $\psi^X_{min}(u)$ ($\psi^X_{max}(u)$) is the increasing function of $X_{(1),n}$ ($X_{(m),n}$), $n \ge 1$. The proof of (3) can be found in Cai and Li (2005a). □

Example 5.1.5. Consider a multivariate compound Poisson model with constant drift functions $\lambda_j = p$, $1 \le j \le m$, Poisson shock arrival process $N = \{N(t), t \ge 0\}$ with rate λ, and damage size vector $(X_{n,1}, \dots, X_{n,m})$. Assume that $(X_{n,1}, \dots, X_{n,m})$ has a joint distribution such that all the one dimensional marginals have an exponential distribution with rate β. Even we know the dependence structure of $(X_{n,1}, \dots, X_{n,m})$, the ruin probabilities $\psi^X_{min}(u)$, $\psi^X_{max}(u)$ and $\psi^X_{sum}(u)$ from (5.1.1), (5.1.2) and (5.1.5) still have no closed formulas. To find computable bounds, we utilize Proposition 5.1.4 and Example 4.1.4. First observe from Example 4.1.4 that

$$(X_{n,1}, \dots, X_{n,m}) \le_{sm} (\underbrace{X_{n,1}, \dots, X_{n,1}}_{m}),$$

where $X_{n,1}$ has an exponential distribution with rate β. It follows from Proposition 5.1.4 that

$$\psi^X_{min}(u) \le \psi_{min}(u),$$
$$\psi^X_{max}(u) \ge \psi_{max}(u),$$
$$\psi^X_{sum}(u) \le \psi_{sum}(u),$$

where $\psi_{min}(u)$, $\psi_{max}(u)$ and $\psi_{sum}(u)$ are the ruin probabilities of type (5.1.1), (5.1.2) and (5.1.5), respectively, with damage size vector $(X_{n,1}, \dots, X_{n,1})$. Obviously, we have

$$\psi_{min}(u) = \psi_{max}(u) = \psi(u),$$

$$\psi_{sum}(u) = \psi(u/m),$$

where $\psi(u)$ is the univariate ruin probability (2.1.9) with Poisson shock arrival process, and exponential distribution damage size. This ruin probability can be calculated explicitly (2.2.1). Thus we have

$$\psi_{min}^{X}(u) \leq \frac{1}{1+\theta} \exp\left(-\frac{\theta\beta}{1+\theta}u\right),$$

$$\psi_{max}^{X}(u) \geq \frac{1}{1+\theta} \exp\left(-\frac{\theta\beta}{1+\theta}u\right),$$

$$\psi_{sum}^{X}(u) \leq \frac{1}{1+\theta} \exp\left(-\frac{\theta\beta}{1+\theta}\frac{u}{m}\right),$$

where $\theta = p\beta/\lambda - 1 > 0$ is known as the relative security loading parameter. □

5.2 Supermodular Dependence

The univariate bounds established in Proposition 5.1.1 hold for any damage size vector $\boldsymbol{X}_n$. If the damage size vector satisfies some positive dependence property, then the product type bounds can be also established. We first review the notion of *supermodular dependence*, which can be found, for example, in Tong (1980) and in Müller and Stoyan (2002).

Definition 5.2.1. Let $\boldsymbol{X} = (X_1, \ldots, X_m)$ be a real random vector.

1. $\boldsymbol{X}$ is said to be positively upper orthant dependent if

$$P(X_1 > x_1, \ldots, X_m > x_m) \geq \prod_{i=1}^{m} P(X_i > x_i).$$

2. $\boldsymbol{X}$ is said to be positively lower orthant dependent if

$$P(X_1 \leq x_1, \ldots, X_m \leq x_m) \geq \prod_{i=1}^{m} P(X_i \leq x_i).$$

3. $\boldsymbol{X}$ is said to be supermodular dependent if

$$(X_1, \ldots, X_m) \geq_{sm} (X_1^I, \ldots, X_m^I), \tag{5.2.1}$$

where $X_1^I, \ldots, X_m^I$ are independent, and X_j^I and X_j, $1 \leq j \leq m$, have the same marginal distribution.

Note that the supermodular dependence yields both upper and lower orthant dependence, that is, if $\boldsymbol{X}$ is supermodular dependent, then the following lower bounds of product type hold for the joint distribution and survival functions.

$$
\begin{aligned}
P\left(X_1 \leq x_1, \ldots, X_m \leq x_m\right) &\geq \prod_{j=1}^{m} P\left(X_j \leq x_j\right), \\
P\left(X_1 > x_1, \ldots, X_m > x_m\right) &\geq \prod_{j=1}^{m} P\left(X_j > x_j\right).
\end{aligned}
$$

Example 5.2.2. Let $Z_1, \ldots, Z_s$ be independent random variables. Let $\mathbf{e}_k$ denote the vector of 1's with k dimension, $k \geq 1$. It follows from Example 4.1.5 that $(Z_1\mathbf{e}_{m_1}, \ldots, Z_s\mathbf{e}_{m_s})$ is supermodular dependent.

Let $\boldsymbol{X} = (X_1, \ldots, X_m)$ have a normal distribution with a mean vector of μ and a covariance matrix of $\Sigma = (\sigma_{ij})$. Then it follows from Example 4.1.6 that $(|X_1|, \ldots, |X_m|)$ is positively upper orthant dependent. □

Assuming that the event arrival process $N(t)$ is a Poisson process, Cai and Li (2005a) established the product type lower bound for $\psi_{and}(u_1, \ldots, u_m)$, by showing that if the damage size vector $\boldsymbol{X}_n$ is associated, then

$$
\left(\sup_{0 \leq t < \infty} S_1(t), \ldots, \sup_{0 \leq t < \infty} S_m(t)\right) \tag{5.2.2}
$$

is also associated. This result also yields the product type upper bound for $\psi_{or}(u_1, \ldots, u_m)$.

If the damage size vector possesses the supermodular dependence, the following bounds will hold.

Theorem 5.2.3. For the multivariate compound Poisson model with a Poisson event arrival process, constant drift functions λ_j and supermodular dependent claim vector, we have

$$
\prod_{j=1}^{m} \psi_j(u_j) \leq \psi_{and}(u_1, \ldots, u_m) \leq \psi_{or}(u_1, \ldots, u_m) \leq 1 - \prod_{j=1}^{m} (1 - \psi_j(u_j))
$$

for any non-negative $u_1, \ldots, u_m$, where $\psi_j(u_j) = P\left(\sup_{0 \leq t < \infty} S_j(t) > u_j\right)$, $1 \leq j \leq m$.

Proof. Consider (2.1.2) and (2.1.6) with a Poisson event arrival process $N(t)$.

For each damage size vector $\boldsymbol{X}_n = (X_{n,1}, \ldots, X_{n,m})$, let $\boldsymbol{X}_n^I = (X_{n,1}^I, \ldots, X_{n,m}^I)$ be the vector in which $X_{n,1}^I, \ldots, X_{n,m}^I$ are independent, and $X_{n,j}^I$ and $X_{n,j}$ have the same marginal distribution, $1 \leq j \leq m$. Also let

$$\psi^I_{and}(u_1, \ldots, u_m) = P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} \left(S^I_j(t)\right) > u_j \right\}\right),$$

where $S^I_j(t) = \sum_{n=1}^{N(t)} X^I_{n,j} - \lambda_j t$, $1 \leq j \leq m$. Since $\boldsymbol{X}_n$ is supermodular dependent, we have $\boldsymbol{X}_n \geq_{sm} \boldsymbol{X}^I_n$. Thus, from Theorem 4.2.1, we have

$$\psi_{and}(u_1, \ldots, u_m) \geq \psi^I_{and}(u_1, \ldots, u_m).$$

We need to show that $\psi^I_{and}(u_1, \ldots, u_m) \geq \prod_{j=1}^{m} \psi_j(u_j)$. Since $N(t)$ is a Poisson process, then

$$N(t) = \max\left\{ n : \sum_{i=1}^{n} E_i \leq t \right\},$$

where E_i's are i.i.d. exponential random variables with a mean of $1/\lambda$. From the Lorentz's inequality (see, for example, Müller and Stoyan 2002), we have, for any $i \geq 1$,

$$\underbrace{(E_i, \ldots, E_i)}_{m} \geq_{sm} (E_{i,1}, \ldots, E_{i,m}), \tag{5.2.3}$$

where $E_{i,j}$'s are i.i.d. exponential random variables with a mean of $1/\lambda$. For any $1 \leq j \leq m$, let $N_j(t)$ denote a Poisson process with inter-event arrival times $E_{i,j}$, $i \geq 1$. Obviously, Poisson processes $\{N_j(t), t \geq 0\}$, $1 \leq j \leq m$, are independent. Let, for each $1 \leq j \leq m$, $N^k_j(t) = N_j(t)$ given that $E_{i,j} = z_i$, $i \geq k+1$.

Conditioning on $\boldsymbol{X}^I_n = (x_{n,1}, \ldots, x_{n,m})$, $n \geq 1$, and $E_i = z_i$, $i \geq k+1$, $\sup_{0 \leq t < \infty} S_j(t)$, $1 \leq j \leq m$, is a decreasing function of $E_1, \ldots, E_k$. Because of (5.2.3), we invoke Lemma 4.1.2 (3) k times, and obtain that

$$\psi^I_{and}(u_1, \ldots, u_m) \geq P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N^k_j(t)} x_{n,j} - \lambda_j t \right) > u_j \right\}\right).$$

As $k \to \infty$, we obtain that

$$\psi^I_{and}(u_1, \ldots, u_m) \geq P\left(\bigcap_{j=1}^{m} \left\{ \sup_{0 \leq t < \infty} \left(\sum_{n=1}^{N_j(t)} x_{n,j} - \lambda_j t \right) > u_j \right\}\right).$$

Unconditioning on $\boldsymbol{X}_n^I$, $n \geq 1$, we have

$$\begin{aligned}\psi_{and}^I(u_1, \ldots, u_m) &\geq P\left(\bigcap_{j=1}^m \left\{\sup_{0\leq t<\infty}\left(\sum_{n=1}^{N_j(t)} X_{n,j}^I - \lambda_j t\right) > u_j\right\}\right) \\ &= \prod_{j=1}^m P\left(\left\{\sup_{0\leq t<\infty}\left(\sum_{n=1}^{N_j(t)} X_{n,j}^I - \lambda_j t\right) > u_j\right\}\right) = \prod_{j=1}^m \psi_j(u_j).\end{aligned}$$

Hence $\psi_{and}(u_1, \ldots, u_m) \geq \prod_{j=1}^m \psi_j(u_j)$.

To establish the third inequality, we consider

$$\psi_{or}(u_1, \ldots, u_m) = 1 - P\left(\sup_{0\leq t<\infty} S_1(t) \leq u_1, \ldots, \sup_{0\leq t<\infty} S_m(t) \leq u_m\right). \qquad (5.2.4)$$

Since $\boldsymbol{X}_n \geq_{sm} \boldsymbol{X}_n^I$, thus, from Theorem 4.2.1, we have

$$\psi_{or}(u_1, \ldots, u_m) \leq \psi_{or}^I(u_1, \ldots, u_m),$$

where

$$\psi_{or}^I(u_1, \ldots, u_m) = P\left(\bigcup_{j=1}^m \left\{\sup_{0\leq t<\infty}\left(S_j^I(t)\right) > u_j\right\}\right).$$

Thus we have

$$P\left(\bigcap_{j=1}^m \left\{\sup_{0\leq t<\infty}\left(S_j(t)\right) \leq u_j\right\}\right) \geq P\left(\bigcap_{j=1}^m \left\{\sup_{0\leq t<\infty}\left(S_j^I(t)\right) \leq u_j\right\}\right).$$

We need to show that

$$P\left(\bigcap_{j=1}^m \left\{\sup_{0\leq t<\infty}\left(S_j^I(t)\right) \leq u_j\right\}\right) \geq \prod_{j=1}^m (1 - \psi_j(u_j)).$$

Since $N(t)$ is a Poisson process, then

$$N(t) = \max\left\{n : \sum_{i=1}^n E_i \leq t\right\},$$

where E_i's are i.i.d. exponential random variables with a mean of $1/\lambda$. From the Lorentz's inequality (see, for example, Müller and Stoyan 2002), we have, for any $i \geq 1$,

$$\underbrace{(E_i, \ldots, E_i)}_{m} \geq_{sm} (E_{i,1}, \ldots, E_{i,m}), \tag{5.2.5}$$

where $E_{i,j}$'s are i.i.d. exponential random variables with a mean of $1/\lambda$. For any $1 \leq j \leq m$, let $N_j(t)$ denote a Poisson process with inter-event arrival times $E_{i,j}$, $i \geq 1$. Obviously, Poisson processes $\{N_j(t), t \geq 0\}$, $1 \leq j \leq m$, are independent. Let, for each $1 \leq j \leq m$, $N_j^k(t) = N_j(t)$ given that $E_{i,j} = z_i$, $i \geq k+1$.

Conditioning on $\boldsymbol{X}_n^I = (x_{n,1}, \ldots, x_{n,m})$, $n \geq 1$, and $E_i = z_i$, $i \geq k+1$, $\sup_{0 \leq t < \infty} S_j(t)$, $1 \leq j \leq m$, is a decreasing function of $E_1, \ldots, E_k$. Because of (5.2.5), we invoke Lemma 4.1.2 (3) k times, and obtain that

$$P\left(\bigcap_{j=1}^{m}\left\{\sup_{0 \leq t < \infty}\left(S_j^I(t)\right) \leq u_j\right\}\right) \geq P\left(\bigcap_{j=1}^{m}\left\{\sup_{0 \leq t < \infty}\left(\sum_{n=1}^{N_j^k(t)} x_{n,j} - \lambda_j t\right) \leq u_j\right\}\right).$$

As $k \to \infty$, we obtain that

$$P\left(\bigcap_{j=1}^{m}\left\{\sup_{0 \leq t < \infty}\left(S_j^I(t)\right) \leq u_j\right\}\right) \geq P\left(\bigcap_{j=1}^{m}\left\{\sup_{0 \leq t < \infty}\left(\sum_{n=1}^{N_j(t)} x_{n,j} - \lambda_j t\right) \leq u_j\right\}\right).$$

Unconditioning on $\boldsymbol{X}_n^I$, $n \geq 1$, we have

$$\begin{aligned} P\left(\bigcap_{j=1}^{m}\left\{\sup_{0 \leq t < \infty}\left(S_j^I(t)\right) \leq u_j\right\}\right) &\geq P\left(\bigcap_{j=1}^{m}\left\{\sup_{0 \leq t < \infty}\left(\sum_{n=1}^{N_j(t)} X_{n,j}^I - \lambda_j t\right) \leq u_j\right\}\right) \\ &= \prod_{j=1}^{m} P\left(\left\{\sup_{0 \leq t < \infty}\left(\sum_{n=1}^{N_j(t)} X_{n,j}^I - \lambda_j t\right) \leq u_j\right\}\right) \\ &= \prod_{j=1}^{m}(1 - \psi_j(u_j)). \end{aligned}$$

Hence $\psi_{or}(u_1, \ldots, u_m) \leq \prod_{j=1}^{m}(1 - \psi_j(u_j))$. □

Example 5.2.4. Consider a multivariate compound Poisson model with constant drift functions λ_j, $1 \leq j \leq m$, Poisson shock arrival process $N = \{N(t), t \geq 0\}$ with rate λ, and damage size vector $(X_{n,1}, \ldots, X_{n,m})$ that is supermodular dependent. Assume that $(X_{n,1}, \ldots, X_{n,m})$ has a joint distribution such that all the one dimensional marginals have an exponential distribution with rate β. Even though we know that $(X_{n,1}, \dots, X_{n,m})$ is supermodular dependent, the ruin probabilities $\psi^X_{and}(u_1, \dots, u_m)$, and $\psi^X_{or}(u_1, \dots, u_m)$ from (2.1.6), and (2.1.7) have no closed formulas. To find computable bounds, it follows from Theorem 5.2.3 and (2.2.1) that we have

$$\prod_{j=1}^{m} \frac{1}{1+\theta_j} \exp\left(-\frac{\theta_j \beta}{1+\theta_j} u_j\right) \le \psi_{and}(u_1, \dots, u_m)$$

$$\le \psi_{or}(u_1, \dots, u_m) \le 1 - \prod_{j=1}^{m} \left(1 - \frac{1}{1+\theta_j} \exp\left(-\frac{\theta_j \beta}{1+\theta_j} u_j\right)\right),$$

where $\theta_j = \lambda_j \beta / \lambda - 1 > 0$, $1 \le j \le m$, are the relative security loading parameter. □

5.3 Multivariate Phase Type Distribution

To calculate these bounds explicitly, we utilize the multivariate phase type distribution to model the damage size vector. Let $\{X(t), t \ge 0\}$ be a right-continuous, continuous-time Markov chain on a finite state space $\mathcal{E}$ with generator Q. Let $\mathcal{E}_i$, $i = 1, \dots, m$, be m nonempty stochastically closed subsets of $\mathcal{E}$ such that $\cap_{i=1}^m \mathcal{E}_i$ is a proper subset of $\mathcal{E}$ (A subset of the state space is said to be stochastically closed if once the process $\{X(t), t \ge 0\}$ enters it, $\{X(t), t \ge 0\}$ never leaves). We assume that absorption into $\cap_{i=1}^m \mathcal{E}_i$ is certain. Since we are interested in the process only until it is absorbed into $\cap_{i=1}^m \mathcal{E}_i$, we may assume, without loss of generality, that $\cap_{i=1}^m \mathcal{E}_i$ consists of one state, which we shall denote by Δ. Thus, without loss of generality, we may write $\mathcal{E} = (\cup_{i=1}^m \mathcal{E}_i) \cup \mathcal{E}_0$ for some subset $\mathcal{E}_0 \subset \mathcal{E}$ with $\mathcal{E}_0 \cap \mathcal{E}_j = \emptyset$ for $1 \le j \le m$. The states in $\mathcal{E}$ are enumerated in such a way that Δ is the first element of $\mathcal{E}$. Thus, the generator of the chain has the form

$$Q = \begin{bmatrix} 0 & \mathbf{0} \\ -A\boldsymbol{e} & A \end{bmatrix}, \tag{5.3.1}$$

where $\mathbf{0} = (0, ..., 0)$ is the d-dimensional row vector of zeros, $\boldsymbol{e} = (1, ..., 1)^T$ is the d-dimensional column vector of 1's, sub-generator A is a $d \times d$ nonsingular matrix, and $d = |\mathcal{E}| - 1$. Let $\boldsymbol{\beta} = (0, \boldsymbol{\alpha})$ be an initial probability vector on $\mathcal{E}$ such that $\boldsymbol{\beta}(\Delta) = 0$.

We define

$$X_i = \inf\{t \ge 0 : X(t) \in \mathcal{E}_i\}, \quad i = 1, \dots, m. \tag{5.3.2}$$

As in Assaf et al. (1984), for simplicity, we shall assume that $P(X_1 > 0, \dots, X_m >$

0) = 1, which means that the underlying Markov chain $\{X(t), t \geq 0\}$ starts within $\mathcal{E}_0$ almost surely. The joint distribution of $(X_1, \ldots, X_m)$ is called a *multivariate phase type* distribution (MPH) with representation $(\boldsymbol{\alpha}, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_m)$, and $(X_1, \ldots, X_m)$ is called a phase type random vector.

When $m = 1$, the distribution of (5.3.2) reduces to the univariate PH distribution introduced in Neuts (1981) (See Chapter 1). Examples of MPH distributions include, among many others, the well-known Marshall-Olkin distribution (Marshall and Olkin 1967). Another example involves multivariate extensions of the Freund distribution (Freund 1961).

Example 5.3.1. Consider a system of s components, $C_1, \ldots, C_s$, subjected to a shock environment. Let T_i denote the lifetime of component C_i, $i = 1, \ldots, s$. A component fails when it receives a fatal shock from the random environment. As long as all the components are functioning, the shock arrival process to component C_i is a Poisson process with rate $\alpha_i^{(0)}$, $i = 1, \ldots, s$, and these shock arrival processes are independent. Upon the first component failure, say C_k, the shock arrival processes to the functioning components C_i, $i \neq k$, change to independent Poisson processes with rates $\alpha_i^{(1)}$, respectively. Upon the j-th component failure, $j \geq 1$, the shock arrival processes to the remaining components that are still functioning change to independent Poisson processes with rates $\alpha_i^{(j)}$, $1 \leq i \leq s$, respectively. Clearly, the lifetime vector $(T_1, \ldots, T_s)$ is dependent. When $s = 2$, the joint distribution of the lifetime vector is a bivariate extension of exponential distribution introduced by Freund (1961).

Let $\mathcal{E} = \{e^K, K \subseteq E\}$ where $E = \{1, \ldots, s\}$ is the index set of the components and e^K denotes the s-dimensional vector with the i-th component being 1 if $i \in K$ and zero otherwise. Let $X = \{X(t), t \geq 0\}$ be a Markov chain with state space $\mathcal{E}$, starting at $e^{\emptyset}$, such that e^E is the absorbing state and the transition rates are given by:

$$Q_{e^K, e^L} = \begin{cases} \alpha_j^{(k)} & \text{if } |K| = k \text{ and } e^L = e^K + e^{\{j\}} \text{ where } j \in K^c \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathcal{E}_i = \{e^K : i \in K\}$, $i = 1, \ldots, s$. Since e^E is the absorbing state, $\mathcal{E}_i$, $i = 1, \ldots, s$, are all stochastically closed. Clearly the lifetimes of components,

$$T_i = \inf\{t \geq 0 : X(t) \in \mathcal{E}_i\}, \quad i = 1, \ldots, s.$$

Thus $(T_1, \ldots, T_s)$ has a multivariate phase type distribution. □

The next example deals with a multivariate extension of the Gamma distribution introduced by Becker and Roux (1981). For the sake of simplifying notations, we discuss only the bivariate case.

Example 5.3.2. Let T_1 and T_2 denote the lifetimes of the components C_1 and C_2 in a two component system where C_1 fails after receiving h shocks and C_2 fails after receiving l shocks, and while both components are still functioning these shocks are governed by two independent Poisson processes with rates α and β respectively. But when one fails, the changed load on the remaining component results in a change of the rate of the Poisson process governing the shocks to the still functioning component; that is, α (β) changes to α' (β') when C_2 (C_1) fails first.

Let $\mathcal{E} = \{(i,j) : 0 \le i \le h, 0 \le j \le l$, and both i and j are integers$\}$. Let $X = \{X(t), t \ge 0\}$ be a Markov chain with state space $\mathcal{E}$, starting at $(0,0)$, such that (h,l) is the absorbing state and the transition rates are given by:

$$Q_{(i,j),(i,j+1)} = \beta \text{ for } 0 \le i < h, 0 \le j < l, \text{ and } Q_{(h,j),(h,j+1)} = \beta' \text{ for } 0 \le j < l, \text{ and}$$

$$Q_{(i,j),(i+1,j)} = \alpha \text{ for } 0 \le i < h, 0 \le j < l, \text{ and } Q_{(i,l),(i+1,l)} = \alpha' \text{ for } 0 \le i < h,$$

and zero otherwise. Let $\mathcal{E}_1 = \{(h,j), 0 \le j \le l\}$, and $\mathcal{E}_2 = \{(i,l), 0 \le i \le h\}$. Since (h,l) is the absorbing state, both $\mathcal{E}_1$ and $\mathcal{E}_2$ are stochastically closed. Clearly, the lifetimes

$$T_i = \inf\{t \ge 0 : X(t) \in \mathcal{E}_i\}, \quad i = 1, 2.$$

Thus (T_1, T_2) has a bivariate phase type distribution. □

The MPH distributions, their properties, and some related applications in reliability theory were discussed in Assaf et al. (1984). As in the univariate case, those MPH distributions (and their densities, Laplace transforms and moments) can be written in a closed form. The set of m-dimensional MPH distributions is dense in the set of all distributions on $[0,\infty)^m$. It is also shown in Assaf et al. (1984) and in Kulkarni (1989) that MPH distributions are closed under marginalization, finite mixture, convolution, and the formation of coherent systems. The following lemma, taken from Cai and Li (2005b), presents the phase type representations of some closure properties.

Lemma 5.3.3. Let $(X_1, \ldots, X_m)$ be of phase type with representation $(\boldsymbol{\alpha}, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_m)$, where $A = (a_{i,j})$. For any $S \subseteq \mathcal{E} - \{\Delta\}$, let A_S denote the sub-matrix of A consisting of all the transition rates from S to S, and $\boldsymbol{\alpha}_S$ is the sub-vector of $\boldsymbol{\alpha}$ consisting all the probability entries on S. Then

1. X_j is of phase type with representation $\left(\frac{\boldsymbol{\alpha}_{\mathcal{E}-\mathcal{E}_j}}{\boldsymbol{\alpha}_{\mathcal{E}-\mathcal{E}_j}\boldsymbol{e}}, A_{\mathcal{E}-\mathcal{E}_j}, |\mathcal{E}-\mathcal{E}_j|\right)$.

2. $X_{(1)} = \min\{X_1, \ldots, X_m\}$ is of phase type with representation $\left(\frac{\boldsymbol{\alpha}_{\mathcal{E}_0}}{\boldsymbol{\alpha}_{\mathcal{E}_0}\boldsymbol{e}},\ A_{\mathcal{E}_0},\ |\mathcal{E}_0|\right)$.

3. $X_{(n)} = \max\{X_1, \ldots, X_m\}$ is of phase type with representation $(\boldsymbol{\alpha},\ A,\ |\mathcal{E}|-1)$.

4. $\sum_{i=1}^n X_i$ has a phase type distribution with representation $(\boldsymbol{\alpha}, T, |\mathcal{E}|-1)$, where $T = (t_{i,j})$ is given by,
$$t_{i,j} = \frac{a_{i,j}}{k(i)}, \tag{5.3.3}$$
and $k(i) =$ number of indexes in $\{j : i \notin \mathcal{E}_j, 1 \le j \le m\}$.

With help from Lemma 5.3.3 and (2.2.2), we obtain the explicit expressions of all the bounds in Propositions 5.1.1 and 5.2.3 as follows.

Proposition 5.3.4. Consider the multivariate compound Poisson model (2.1.2) with constant drift rates p_j, a Poisson event arrival process of rate λ, and phase type distributed damage size vectors with representation $(\boldsymbol{\alpha}, A, \mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_m)$, where $A = (a_{i,j})$.

1. $\psi_j(u_j) = -\frac{\lambda}{p_j}\frac{\boldsymbol{\alpha}_{\mathcal{E}-\mathcal{E}_j}}{\boldsymbol{\alpha}_{\mathcal{E}-\mathcal{E}_j}\boldsymbol{e}}A^{-1}_{\mathcal{E}-\mathcal{E}_j}\exp\left\{\left(A_{\mathcal{E}-\mathcal{E}_j} - \frac{\lambda}{p_j}\boldsymbol{t}_0\frac{\boldsymbol{\alpha}_{\mathcal{E}-\mathcal{E}_j}}{\boldsymbol{\alpha}_{\mathcal{E}-\mathcal{E}_j}\boldsymbol{e}}A^{-1}_{\mathcal{E}-\mathcal{E}_j}\right)u_j\right\}\boldsymbol{e}$, where $\boldsymbol{t}_0 = -A_{\mathcal{E}-\mathcal{E}_j}\boldsymbol{e}$.

2. $\psi_{min}(u_{(m)}) = -\frac{\lambda}{p_{(m)}}\frac{\boldsymbol{\alpha}_{\mathcal{E}_0}}{\boldsymbol{\alpha}_{\mathcal{E}_0}\boldsymbol{e}}A^{-1}_{\mathcal{E}_0}\exp\left\{\left(A_{\mathcal{E}_0} - \frac{\lambda}{p_{(m)}}\boldsymbol{t}_0\frac{\boldsymbol{\alpha}_{\mathcal{E}_0}}{\boldsymbol{\alpha}_{\mathcal{E}_0}\boldsymbol{e}}A^{-1}_{\mathcal{E}_0}\right)u_{(m)}\right\}\boldsymbol{e}$, where $\boldsymbol{t}_0 = -A_{\mathcal{E}_0}\boldsymbol{e}$ and $p_{(m)} = \max\{p_1, \ldots, p_m\}$.

3. $\psi_{max}(u_{(1)}) = -\frac{\lambda}{p_{(1)}}\boldsymbol{\alpha}A^{-1}\exp\left\{\left(A - \frac{\lambda}{p_{(1)}}\boldsymbol{t}_0\boldsymbol{\alpha}A^{-1}\right)u_{(1)}\right\}\boldsymbol{e}$, where $\boldsymbol{t}_0 = -A\boldsymbol{e}$ and $p_{(1)} = \min\{p_1, \ldots, p_m\}$.

4. $\psi_{sum}(\sum_{j=1}^m u_j) = -\frac{\lambda}{\sum_{j=1}^m p_j}\boldsymbol{\alpha}T^{-1}\exp\left\{\left(T - \frac{\lambda}{\sum_{j=1}^m p_j}\boldsymbol{t}_0\boldsymbol{\alpha}T^{-1}\right)\left(\sum_{j=1}^m u_j\right)\right\}\boldsymbol{e}$, where $\boldsymbol{t}_0 = -T\boldsymbol{e}$, and T is defined as in (5.3.3).

5.4 Multivariate Compound Poisson Process Models with Marshall-Olkin Distributed Damages

In this section, we illustrate our results using the multivariate Marshall-Olkin distribution, and also show some interesting effects of different parameters on the bounds.

Let $\{E_S, S \subseteq \{1, \ldots, m\}\}$ be a sequence of independent, exponentially distributed random variables, with E_S having mean $1/\lambda_S$. Let

$$X_j = \min\{E_S : S \ni j\}, \quad j = 1, \ldots, m. \tag{5.4.1}$$

The joint distribution of $(X_1, \ldots, X_m)$ is called the Marshall-Olkin distribution with parameters $\{\lambda_S, S \subseteq \{1, \ldots, m\}\}$ (Marshall and Olkin 1967). In the reliability context, $X_1, \ldots, X_m$ can be viewed as the lifetimes of m components operating in a random shock environment where a fatal shock governed by Poisson process $\{N_S(t), t \geq 0\}$ with rate λ_S destroys all the components with indexes in $S \subseteq \{1, \ldots, m\}$ simultaneously. Assume that these Poisson shock arrival processes are independent, then,

$$X_j = \inf\{t : N_S(t) \geq 1,\, S \ni j\}, \quad j = 1, \ldots, m. \tag{5.4.2}$$

Let $\{\mathcal{M}_S(t), t \geq 0\}$, $S \subseteq \{1, \ldots, m\}$, be independent Markov chains with absorbing state Δ_S, each representing the exponential distribution with parameter λ_S. It follows from (5.4.2) that $(X_1, \ldots, X_m)$ is of phase type with the underlying Markov chain on the product space of these independent Markov chains with absorbing classes $\mathcal{E}_j = \{(e_S) : e_S = \Delta_S \text{ for some } S \ni j\}$, $1 \leq j \leq m$. It is also easy to verify that the marginal distribution of the j-th component of the Marshall-Olkin distributed random vector is exponential with mean $1/\sum_{S:S\ni j} \lambda_S$.

Consider a multivariate compound Poisson model with constant drift functions $\lambda_j(n) = p_j$, $1 \leq j \leq m$, Poisson shock arrival process $N = \{N(t), t \geq 0\}$ with rate λ, and damage size vector $(X_{n,1}, \ldots, X_{n,m})$ that has a Marshall-Olkin distribution with parameters $\{\lambda_S, S \subseteq \{1, \ldots, m\}\}$. It follows from (5.4.1) that any Marshall-Olkin distribution is supermodular dependent. Thus, from Proposition 5.2.3, we have

$$\left(\prod_{j=1}^{m} \frac{1}{1+\theta_j}\right) \exp\left(-\sum_{j=1}^{m} \left\{\frac{\theta_j}{1+\theta_j}\left(\sum_{S:S\ni j} \lambda_S\right) u_j\right\}\right) \leq \psi_{and}(u_1, \ldots, u_m),$$

$$\psi_{or}(u_1, \ldots, u_m) \leq 1 - \prod_{j=1}^{m} \left[1 - \frac{1}{1+\theta_j} \exp\left(-\left\{\frac{\theta_j}{1+\theta_j}\left(\sum_{S:S\ni j} \lambda_S\right) u_j\right\}\right)\right],$$

for any non-negative $u_1, \ldots, u_m$, where the relative security loading $\theta_j = \left(\sum_{S:S\ni j} \lambda_S\right) p_j/\lambda - 1$, $1 \leq j \leq m$.

To calculate the other bounds, we need to simplify the underlying Markov chain for the

Marshall-Olkin distribution and obtain its phase type representation. Let $\{X(t), t \geq 0\}$ be a Markov chain with state space $\mathcal{E} = \{S : S \subseteq \{1, \ldots, m\}\} = \{\Delta, e_1, ..., e_d\}$, and starting at $\emptyset$ almost surely. The index set $\{1, \ldots, m\}$ is the absorbing state Δ, and $\mathcal{E}_0 = \{\emptyset\}$, $\mathcal{E}_j = \{S : S \ni j\}, \quad j = 1, \ldots, m.$

It follows from (5.4.2) that its sub-generator is given by $A = (a_{i,j})$, where

$$a_{i,j} = \sum_{L:\ L \subseteq S^*,\, L \cup S = S^*} \lambda_L, \text{ if } i = S,\, j = S^* \text{ and } S \subset S^*,$$

$$a_{i,i} = \sum_{L:\ L \subseteq S} \lambda_L - \Lambda, \text{ if } i = S \text{ and } \Lambda = \sum_S \lambda_S,$$

and zero otherwise. Using the results in Chapters 3-4 and these parameters, we can calculate the bounds. To illustrate the results, we consider the bivariate and trivariate cases.

5.4.1 Bivariate Case

When $m = 2$, the state space $\mathcal{E} = \{12, 2, 1, \emptyset\}$ and $\mathcal{E}_j = \{12, j\}$, $j = 1, 2$, where 12 is the absorbing state. The initial probability vector is $(0, 0, 0, 1)$, and its sub-generator is given by

$$A = \begin{bmatrix} -\lambda_{12} - \lambda_1 & 0 & 0 \\ 0 & -\lambda_{12} - \lambda_2 & 0 \\ \lambda_2 & \lambda_1 & -\Lambda + \lambda_\emptyset \end{bmatrix}, \tag{5.4.3}$$

where $\Lambda = \lambda_{12} + \lambda_2 + \lambda_1 + \lambda_\emptyset$. The matrix T in Proposition 5.3.4 is given by

$$T = \begin{bmatrix} -\lambda_1 - \lambda_{12} & 0 & 0 \\ 0 & -\lambda_2 - \lambda_{12} & 0 \\ \dfrac{\lambda_2}{2} & \dfrac{\lambda_1}{2} & -\dfrac{\Lambda_0}{2} \end{bmatrix}, \tag{5.4.4}$$

where $\Lambda_0 = \lambda_1 + \lambda_2 + \lambda_{12} = \Lambda - \lambda_\emptyset$. Since $\mathcal{E} - \mathcal{E}_1 = \{2, \emptyset\}$ and $\mathcal{E} - \mathcal{E}_1 = \{1, \emptyset\}$, then we have

$$A_{\mathcal{E}-\mathcal{E}_1} = \begin{bmatrix} -\lambda_{12} - \lambda_1 & 0 \\ \lambda_2 & -\Lambda + \lambda_\emptyset \end{bmatrix}, \tag{5.4.5}$$

and

$$A_{\mathcal{E}-\mathcal{E}_2} = \begin{bmatrix} -\lambda_{12} - \lambda_2 & 0 \\ \lambda_1 & -\Lambda + \lambda_\emptyset \end{bmatrix}. \tag{5.4.6}$$

To study the effect of dependence on the bounds, we calculate $\psi_{min}(u_{(m)})$, $\psi_{max}(u_{(1)})$, $\psi_{sum}(u_1+u_2)$ and the product type bounds in Proposition 5.2.3, respectively, under several different sets of model parameters in the following example.

Example 5.4.1. Assume that $\lambda = 1.6$ and $p_1 = p_2 = 3$. Let ρ be the correlation coefficient between the damage vector $(X_{n,1}, X_{n,2})$. Then, it is not hard to find that

$$\rho = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}. \tag{5.4.7}$$

Note that none of ρ and the matrix A in (5.4.4) involves $\lambda_\emptyset$. We introduce $\lambda_\emptyset$ in the model because we want to change the model parameters in a systematic fashion according to supermodular order, so that the effect of claim dependence on the ruin probabilities can be investigated.

We consider the following three cases of the damage vector $(X_{n,1}, X_{n,2})$. The correlation coefficients in the three cases are increasing, which indicates the increasing (linear) dependence of the damage vector in the three cases. In fact, it follows from Li and Xu (2000) that the damage size vector in Case 1 is less dependent than that in Case 2, which, in turn, is less dependent than that in Case 3, all in supermodular order. The analytic forms of these bounds in the three cases and the numerical values in Tables 5.1 and 5.2 were easily produced by Mathematica by using the formulas given in Proposition 5.3.4.

The first column of the Tables lists several values of u_1 and u_2, and the next several columns list values of these bounds in the following three cases.

Case 1 - independent damage vector: Let $\lambda_{12} = 0$, $\lambda_1 = 1.15$, $\lambda_2 = 1.17$, $\lambda_\emptyset = 0$. Then the damage vector $(X_{n,1}, X_{n,2})$ is independent with $\rho = 0$ and

$$\begin{aligned} \psi_{min}(u) &= 0.229885\, e^{-1.7866u}, \\ \psi_{sum}(2u) &= 0.516615\, e^{-0.909286u} - 0.0568111\, e^{-3.19738u}, \\ \psi_{max}(u) &= 0.712874\, e^{-0.31756u} + 0.0000250601\, e^{-1.16014u} - 0.0231758\, e^{-2.62897u}, \\ \psi_1(u) &= 0.463768\, e^{-0.616667u}, \;\; \psi_2(u) = 0.455840\, e^{-0.636667u}. \end{aligned}$$

Note that even the damage sizes are independent, but the damage processes are still positively dependent.

Case 2 - slightly dependent damage vector: Let $\lambda_{12} = 0.18$, $\lambda_1 = 0.97$, $\lambda_2 = 0.99$, $\lambda_\emptyset = 0.18$. Then the damage vector $(X_{n,1}, X_{n,2})$ is slightly dependent with $\rho = 0.0841$. and

$$\begin{aligned}
\psi_{min}(u) &= 0.249221\, e^{-1.60667u}, \\
\psi_{sum}(2u) &= 0.513675\, e^{-0.880331u} - 0.0000160242\, e^{-2.31993u} - 0.0538543\, e^{-3.0464u}, \\
\psi_{max}(u) &= 0.694215\, e^{-0.336674u} + 0.0000266203\, e^{-1.16014u} - 0.0238541\, e^{-2.42985u}, \\
\psi_1(u) &= 0.463768\, e^{-0.616667u},\ \psi_2(u) = 0.455840\, e^{-0.636667u}.
\end{aligned}$$

Case 3 - highly dependent vector: Let $\lambda_{12} = 1.1$, $\lambda_1 = 0.05$, $\lambda_2 = 0.07$, $\lambda_\emptyset = 1.1$. Then the damage vector $(X_{n,1}, X_{n,2})$ is highly dependent with $\rho = 0.9016$ and

$$\begin{aligned}
\psi_{min}(u) &= 0.437158\, e^{-0.686667u}, \\
\psi_{sum}(2u) &= 0.465657\, e^{-0.6483u} - 0.00111286\, e^{-2.31406u} - 0.0047396\, e^{-2.36431u}, \\
\psi_{max}(u) &= 0.485566\, e^{-0.576616u} + 0.000041294\, e^{-1.16093u} - 0.00315671\, e^{-1.26912u}, \\
\psi_1(u) &= 0.463768\, e^{-0.616667u},\ \psi_2(u) = 0.455840\, e^{-0.636667u}.
\end{aligned}$$

In all the three cases, the marginal distributions of $X_{n,1}$ and $X_{n,2}$ are the same. Indeed, $X_{n,1}$ and $X_{n,2}$ have exponential distributions with means $1/(\lambda_1 + \lambda_{12}) = 1/1.15$ and $1/(\lambda_2 + \lambda_{12}) = 1/1.17$, respectively.

The product type bounds in Proposition 5.2.3 are the functions of the ruin probabilities $\psi_1(u)$ and $\psi_2(u)$, which do not depend on the dependence structure of the damage vector $(X_{n,1}, X_{n,2})$. Since these bounds are obtained for *independent damage processes*, the bounds in Proposition 5.2.3 should out-perform (under-perform) those in Proposition 5.1.1 when the damage vector $(X_{n,1}, X_{n,2})$ is slightly (highly) dependent. Indeed, the tables show that the bounds in Proposition 5.2.3 are better than those in Proposition 5.1.1 in Cases 1 and 2 for slightly dependent damage vectors while the bounds in Proposition 5.1.1 are better than those in Proposition 5.2.3 in Case 3 for highly dependent damage vectors. Note, however, that the bounds in Proposition 5.2.3 are not sharp for *independent damage vectors*.

The tables also show that, serving as lower and upper bounds for $\psi_{sim}(u_1, u_2)$, the lower bound $\psi_{min}(u_{(2)})$ and the upper bound $\psi_{sum}(u_1 + u_2)$ are tighter in Case 3 than in Cases 1 and 2. Similarly, serving as lower and upper bounds for $\psi_{or}(u_1, u_2)$, the lower bound $\psi_{sum}(u_1+u_2)$ and the upper bound $\psi_{max}(u_{(1)})$ are tighter in Case 3 than in Cases 1 and 2. Indeed, in the extremal case or the comonotone case where $X_{n,1} = X_{n,2}$, we have $\psi_{min}(u_{(2)}) = \psi_{sim}(u_1, u_2) = \psi_{sum}(u_1+u_2) = \psi_{and}(u_1, u_2) = \psi_{or}(u_1, u_2) = \psi_{max}(u_{(1)})$. This further indicates that the bounds in Proposition 5.1.1 are better for highly dependent damage vectors.

In addition, as proved in Proposition 5.1.4, the tables display that $\psi_{min}(u)$ ($\psi_{sum}(u)$) and $\psi_{max}(u)$ have opposite monotonicity properties when dependence among the damage sizes increases. □

Table 5.1: Effects of dependence on the bounds for $\psi_{sim}(u_1, u_2)$.

		$\psi_{min}(u_{(2)})$			$\prod_{j=1}^{2}\psi_j(u_j)$	$\psi_{sum}(u_1+u_2)$		
u_1	u_2	Case 1	Case 2	Case 3	Cases 1-3	Case 1	Case 2	Case 3
0.5	0.5	0.09409	0.11161	0.31012	0.11297	0.31640	0.31903	0.33493
1.0	1.0	0.03851	0.04998	0.22000	0.06037	0.20578	0.21043	0.24295
1.5	1.5	0.01576	0.02238	0.15607	0.03226	0.13161	0.13659	0.17592
2.0	2.0	0.00645	0.01002	0.11072	0.01724	0.08373	0.08819	0.12729
2.5	2.5	0.00264	0.00449	0.07854	0.00921	0.05318	0.05684	0.09207
3.0	3.0	0.00108	0.00201	0.05572	0.00492	0.03376	0.03661	0.06658

Table 5.2: Effects of dependence on the bounds for $\psi_{or}(u_1, u_2)$.

		$\psi_{sum}(u_1 + u_2)$			$1 - \prod_{j=1}^{2}(1 - \psi_j(u_j))$	$\psi_{max}(u_{(1)})$		
u_1	u_2	Case 1	Case 2	Case 3	Cases 1-3	Case 1	Case 2	Case 3
0.5	0.5	0.31640	0.31903	0.33493	0.55931	0.60200	0.57960	0.36230
1.0	1.0	0.20578	0.21043	0.24295	0.43111	0.51725	0.49368	0.27192
1.5	1.5	0.13161	0.13659	0.17592	0.32705	0.44229	0.41834	0.20400
2.0	2.0	0.08373	0.08819	0.12729	0.24546	0.37761	0.35387	0.15301
2.5	2.5	0.05318	0.05684	0.09207	0.18285	0.32224	0.29914	0.11474
3.0	3.0	0.03376	0.03661	0.06658	0.13550	0.27495	0.25283	0.08603

5.4.2 Trivariate Case

The trivariate case exhibits more interesting patterns. When $m = 3$, the state space $\mathcal{E} = \{123, 23, 13, 12, 3, 2, 1, \emptyset\}$ with the absorbing state $\Delta = 123$ and stochastically closed subsets

$$\mathcal{E}_1 = \{123, 13, 12, 1\},\ \mathcal{E}_2 = \{123, 23, 12, 2\},\ \mathcal{E}_3 = \{123, 23, 13, 3\}, \tag{5.4.8}$$

and

$$\mathcal{E} - \mathcal{E}_1 = \{23, 3, 2, \emptyset\},\ \mathcal{E} - \mathcal{E}_2 = \{13, 3, 1, \emptyset\},\ \mathcal{E} - \mathcal{E}_3 = \{12, 2, 1, \emptyset\}. \tag{5.4.9}$$

The initial probability vector is $(0, 0, 0, 0, 0, 0, 0, 1)$, and its sub-generator is given by

$$A = \begin{bmatrix} \bar{\Lambda}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\Lambda}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\Lambda}_3 & 0 & 0 & 0 & 0 \\ \lambda_{23} + \lambda_2 & \lambda_{13} + \lambda_1 & 0 & \bar{\Lambda}_4 & 0 & 0 & 0 \\ \lambda_{23} + \lambda_3 & 0 & \lambda_{12} + \lambda_1 & 0 & \bar{\Lambda}_5 & 0 & 0 \\ 0 & \lambda_{13} + \lambda_3 & \lambda_{12} + \lambda_2 & 0 & 0 & \bar{\Lambda}_6 & 0 \\ \lambda_{23} & \lambda_{13} & \lambda_{12} & \lambda_3 & \lambda_2 & \lambda_1 & \bar{\Lambda}_7 \end{bmatrix};$$

the matrix T in Proposition 5.3.4 is given by

$$T = \begin{bmatrix} \bar{\Lambda}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\Lambda}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\Lambda}_3 & 0 & 0 & 0 & 0 \\ \frac{\lambda_{23}+\lambda_2}{2} & \frac{\lambda_{13}+\lambda_1}{2} & 0 & \frac{\bar{\Lambda}_4}{2} & 0 & 0 & 0 \\ \frac{\lambda_{23}+\lambda_3}{2} & 0 & \frac{\lambda_{12}+\lambda_1}{2} & 0 & \frac{\bar{\Lambda}_5}{2} & 0 & 0 \\ 0 & \frac{\lambda_{13}+\lambda_3}{2} & \frac{\lambda_{12}+\lambda_2}{2} & 0 & 0 & \frac{\bar{\Lambda}_6}{2} & 0 \\ \frac{\lambda_{23}}{3} & \frac{\lambda_{13}}{3} & \frac{\lambda_{12}}{3} & \frac{\lambda_3}{3} & \frac{\lambda_2}{3} & \frac{\lambda_1}{3} & \frac{\bar{\Lambda}_7}{3} \end{bmatrix};$$

the sub-matrix $A_{\mathcal{E}-\mathcal{E}_1}$ is given by:

$$A_{\mathcal{E}-\mathcal{E}_1} = \begin{bmatrix} \bar{\Lambda}_1 & 0 & 0 & 0 \\ \lambda_{23}+\lambda_2 & \bar{\Lambda}_4 & 0 & 0 \\ \lambda_{23}+\lambda_3 & 0 & \bar{\Lambda}_5 & 0 \\ \lambda_{23} & \lambda_3 & \lambda_2 & \bar{\Lambda}_7 \end{bmatrix};$$

the sub-matrix $A_{\mathcal{E}-\mathcal{E}_2}$ is given by:

$$A_{\mathcal{E}-\mathcal{E}_2} = \begin{bmatrix} \bar{\Lambda}_2 & 0 & 0 & 0 \\ \lambda_{13}+\lambda_1 & \bar{\Lambda}_4 & 0 & 0 \\ \lambda_{13}+\lambda_3 & 0 & \bar{\Lambda}_6 & 0 \\ \lambda_{13} & \lambda_3 & \lambda_1 & \bar{\Lambda}_7 \end{bmatrix};$$

and the sub-matrix $A_{\mathcal{E}-\mathcal{E}_3}$ is given by:

$$A_{\mathcal{E}-\mathcal{E}_3} = \begin{bmatrix} \bar{\Lambda}_3 & 0 & 0 & 0 \\ \lambda_{12}+\lambda_1 & \bar{\Lambda}_5 & 0 & 0 \\ \lambda_{12}+\lambda_2 & 0 & \bar{\Lambda}_6 & 0 \\ \lambda_{12} & \lambda_2 & \lambda_1 & \bar{\Lambda}_7 \end{bmatrix}.$$

where $\Lambda = \lambda_{123} + \lambda_{23} + \lambda_{13} + \lambda_{12} + \lambda_3 + \lambda_2 + \lambda_1 + \lambda_\emptyset$, $\bar{\Lambda}_1 = -\Lambda + \lambda_{23} + \lambda_3 + \lambda_2 + \lambda_\emptyset$, $\bar{\Lambda}_2 = -\Lambda + \lambda_{13} + \lambda_3 + \lambda_1 + \lambda_\emptyset$, $\bar{\Lambda}_3 = -\Lambda + \lambda_{12} + \lambda_2 + \lambda_1 + \lambda_\emptyset$, $\bar{\Lambda}_4 = -\Lambda + \lambda_3 + \lambda_\emptyset$, $\bar{\Lambda}_5 = -\Lambda + \lambda_2 + \lambda_\emptyset$, $\bar{\Lambda}_6 = -\Lambda + \lambda_1 + \lambda_\emptyset$, $\bar{\Lambda}_7 = -\Lambda + \lambda_\emptyset$.

To study the effect of dependence on the bounds, we calculate $\psi_{min}(u_{(m)})$, $\psi_{max}(u_{(1)})$,

$\psi_{sum}(u_1 + u_2 + u_3)$ and the product type bounds in Proposition 5.2.3, respectively, under several different sets of model parameters in the following example.

Example 5.4.2. Assume that $\lambda = 6.6$ and $p_1 = p_2 = p_3 = 3$. Again, we introduce $\lambda_\emptyset$ in the model because we want to change the model parameters in a systematic fashion according to supermodular order, so that the effect of damage dependence on the ruin probabilities can be investigated.

We consider the following three cases of the damage vector $(X_{n,1}, X_{n,2}, X_{3,n})$.

Case 1: Let $\lambda_{123} = 1$, $\lambda_{23} = 0.5$, $\lambda_{13} = 1$, $\lambda_{12} = 1$, $\lambda_3 = 1.5$, $\lambda_2 = 1.5$, $\lambda_1 = 1$, $\lambda_\emptyset = 0.5$.

Case 2: Let $\lambda_{123} = 1$, $\lambda_{23} = 1$, $\lambda_{13} = 1$, $\lambda_{12} = 1$, $\lambda_3 = 1$, $\lambda_2 = 1$, $\lambda_1 = 1$, $\lambda_\emptyset = 1$.

Case 3: Let $\lambda_{123} = 1.5$, $\lambda_{23} = 1$, $\lambda_{13} = 0.5$, $\lambda_{12} = 0.5$, $\lambda_3 = 1$, $\lambda_2 = 1$, $\lambda_1 = 1.5$, $\lambda_\emptyset = 1$.

In all the three cases, the marginal distributions of $X_{n,1}$, $X_{n,2}$ and $X_{3,n}$ are the same. Indeed, $X_{n,1}$, $X_{n,2}$ and $X_{3,n}$ have exponential distributions with means $1/(\lambda_1 + \lambda_{12} + \lambda_{13} + \lambda_{123}) = 1/4$, $1/(\lambda_2 + \lambda_{12} + \lambda_{23} + \lambda_{123}) = 1/4$, and $1/(\lambda_3 + \lambda_{23} + \lambda_{13} + \lambda_{123}) = 1/4$ respectively. It follows from Li and Xu (2000) that the damage size vector in Case 1 is less dependent than that in Case 2, which, in turn, is less dependent than that in Case 3, all in supermodular order.

The first column of the Tables 5.3-5.5 lists several values of u_1, u_2, and u_3 and the next several columns list values of these bounds in the three cases.

The product type bounds in Proposition 5.2.3 are the functions of the ruin probabilities $\psi_1(u)$, $\psi_2(u)$ and $\psi_3(u)$, which do not depend on the dependence structure of the damage vector $(X_{n,1}, X_{n,2}, X_{3,n})$.

In addition, as proved in Proposition 5.1.4, the tables display that $\psi_{min}(u)$ $(\psi_{sum}(u))$ and $\psi_{max}(u)$ have opposite monotonicity properties when dependence among the claim sizes increases.

Table 5.3: Effects of dependence on the bounds for $\psi_{sim}(u_1, u_2, u_3)$.

			$\psi_{min}(u_{(3)})$			$\prod_{j=1}^{3} \psi_j(u_j)$
u_1	u_2	u_3	Case 1	Case 2	Case 3	Cases 1-3
0.5	0.5	0.5	0.020724355831	0.028511356748	0.028511356748	0.011181317182
1.0	1.0	1.0	0.001464200879	0.002586491930	0.002586491930	0.000751446154
1.5	1.5	1.5	0.000103447568	0.000234641254	0.000234641254	0.000050501324
2.0	2.0	2.0	0.000007308696	0.000021286174	0.000021286174	0.000003393967
2.5	2.5	2.5	0.000000516368	0.000001931038	0.000001931038	0.000000228093
3.0	3.0	3.0	0.000000036482	0.000000175180	0.000000175180	0.000000015329

Table 5.4: Effects of dependence on the bounds for $\psi_{sum}(u_1, u_2, u_3)$.

			$\psi_{sum}(u_1 + u_2 + u_3)$			$1 - \prod_{j=1}^{3} (1 - \psi_j(u_j))$
u_1	u_2	u_3	Case 1	Case 2	Case 3	Cases 1-3
0.5	0.5	0.5	0.17067593469	0.17374829838	0.17750744640	0.53201251469
1.0	1.0	1.0	0.04859281746	0.05075942339	0.05297176874	0.24869833360
1.5	1.5	1.5	0.01382231106	0.01481379823	0.01578906700	0.10684080014
2.0	2.0	2.0	0.00393177170	0.00432327251	0.00470613137	0.04441000938
2.5	2.5	2.5	0.00111839686	0.00126170780	0.00140272195	0.01821807798
3.0	3.0	3.0	0.00031812924	0.00036821796	0.00041809901	0.00743386134

Table 5.5: Effects of dependence on the bounds for $\psi_{max}(u_{(1)})$.

			$1-\prod_{j=1}^{3}(1-\psi_j(u_j))$	$\psi_{max}(u_{(1)})$		
u_1	u_2	u_3	Case 1-3	Case 1	Case 2	Cases 3
0.5	0.5	0.5	0.53201251469	0.71981348939	0.70534396690	0.64350538069
1.0	1.0	1.0	0.24869833360	0.58698719541	0.56814126151	0.49074741143
1.5	1.5	1.5	0.10684080014	0.47844155130	0.45738687847	0.37395699468
2.0	2.0	2.0	0.04441000938	0.38996320438	0.36821708707	0.28495229335
2.5	2.5	2.5	0.01821807798	0.31784711420	0.29643126781	0.21713121803
3.0	3.0	3.0	0.00743386134	0.25906748834	0.23864046326	0.16545213408

5.5 Simulation Results

In this section, we develop a simulation procedure for the multivariate compound point process with drifts in which the shock arrival process is not Poisson. We mainly consider the case where the shock arrival process has certain Markov dependence; that is, any interarrival time interval depends on the past only through its prior interarrival time interval. The bivariate exponential and Weibull distributions are used to model such a Markov dependence, and the damage sizes are modeled via Marshall-Olkin distributions. In each case under our study, 5000 independent sample paths are simulated, and various ruin probabilities are estimated. These simulation results well illustrate the monotonicity properties we obtained in Chapter 4.

5.5.1 Generating Point Processes

To generate a point process with bounded conditional intensity, we have used the Shedler-Lewis thinning technique (Daley and Vere-Jones 1988). The Shedler-Lewis thinning technique is one of several techniques that can be carried over to the point process context when the conditional intensity λ^* is known explicitly as a function of past variables. The thinning technique is particularly useful when λ^* is conditionally bounded, by which we mean that for every $n = 1, 2, \dots$ and all sequences $t_1, \dots, t_{n-1}$ with $t_1 < \cdots < t_{n-1} < t$, the hazard function $h_n(\cdot \mid \cdot)$ satisfies

$$h_n(t+u \mid t_1, \dots, t_{n-1}) \le M^*(t) \quad \text{all } u > 0,$$

for some $M^*(t) = M^*(t; t_1, \ldots, t_{n-1}) < \infty$. The Algorithm is as follows:

(1) set $t = 0, i = 1$;
(2) calculate $M^*(t)$;
(3) generate an exponential random variable T with mean $1/M^*(t)$ and a random variable U uniformly distributed on (0,1);
(4) if $\lambda^*(t+T)/M^*(t) > U$, replace t by $t+T$ and return to step (2); while otherwise,
(5) set $t_i = t+T$, advance i by 1, replace t by t_i, and return to step (2).

We have done two cases: one is based on the Bivariate Marshall-Olkin Exponential Distribution(BVE) and the other one is on the Bivariate Weibull Distribution(BVW).

1. The Bivariate Exponential Distribution
   The Bivariate Exponential Distribution has a joint survival probability function of two random variables T_1 and T_2,
   $$\bar{F}(t_1, t_2) = P[T_1 > t_1, T_2 > t_2] = e^{-\lambda_1 t_1 - \lambda_2 t_2 - \lambda_{12} max(t_1, t_2)} \text{ for } t_1 \geq 0, t_2 \geq 0.$$
   The BVE has exponential marginal distributions with survival functions given by:
   $$\bar{F}(t_1) = P[T_1 > t_1] = e^{-(\lambda_1 + \lambda_{12})t_1} \qquad \text{for } t_1 \geq 0,$$
   $$\bar{F}(t_2) = P[T_2 > t_2] = e^{-(\lambda_2 + \lambda_{12})t_2} \qquad \text{for } t_2 \geq 0.$$
   Given that T_1 and T_2 have the BVE distribution, then the conditional survival probability $P[T_2 > t_2 \mid T_1 = t_1]$ is given by
   $$P[T_2 > t_2 \mid T_1 = t_1] = \begin{cases} e^{-\lambda_2 t_2} & \text{for } \quad t_1 > t_2; \\ \frac{\lambda_1}{\lambda_1 + \lambda_{12}} e^{-\lambda_{12}(t_2 - t_1) - \lambda_2 t_2} & \text{for } \quad t_1 \leq t_2. \end{cases}$$
   The conditional hazard function $h(t_2 \mid T_1 = t_1)$ is given by
   $$h(t_2 \mid T_1 = t_1) = \begin{cases} \lambda_2 & \text{for } \quad t_1 > t_2; \\ \lambda_{12} + \lambda_2 & \text{for } \quad t_1 \leq t_2. \end{cases}$$
2. The Bivariate Weibull Distribution
   The Bivariate Weibull Distribution has a joint survival probability function of two random variables T_1 and T_2
   $$\bar{F}(t_1, t_2) = P[T_1 > t_1, T_2 > t_2] = e^{-\lambda_1 t_1^\beta - \lambda_2 t_2^\beta - \lambda_{12} max(t_1^\beta, t_2^\beta)} \text{ for } t_1 \geq 0, t_2 \geq 0.$$
   The BVW has weibull marginal distributions with survival functions given by:
   $$\bar{F}(t_1) = P[T_1 > t_1] = e^{-(\lambda_1 + \lambda_{12})t_1^\beta} \qquad \text{for } t_1 \geq 0,$$

$$\bar{F}(t_2) = P[T_2 > t_2] = e^{-(\lambda_2+\lambda_{12})t_2^\beta} \qquad \text{for } t_2 \geq 0.$$

Given that T_1 and T_2 have the BVW distribution, then the conditional survival probability $P[T_2 > t_2 \mid T_1 = t_1]$ is given by

$$P[T_2 > t_2 \mid T_1 = t_1] = \begin{cases} e^{-\lambda_2 t_2^\beta} & \text{for} \quad t_1 > t_2; \\ \frac{\lambda_1}{\lambda_1+\lambda_{12}} e^{-\lambda_{12}(t_2^\beta - t_1^\beta) - \lambda_2 t_2^\beta} & \text{for} \quad t_1 \leq t_2. \end{cases}$$

The conditional hazard function $h(t_2 \mid T_1 = t_1)$ is given by

$$h(t_2 \mid T_1 = t_1) = \begin{cases} \beta\lambda_2 t_2^{\beta-1} & \text{for} \quad t_1 > t_2; \\ \beta(\lambda_{12} + \lambda_2) t_2^{\beta-1} & \text{for} \quad t_1 \leq t_2. \end{cases}$$

5.5.2 Generating Damage vectors

The algorithm that has been used to generate damage vectors is as follows:

(1) Generate i.i.d random variables $Y_i \sim exp(\lambda_i)$ respectively, where $i \in \{1, 2, 3, 12, 13, 23, 123\}$;
(2) Set $X_j = \min\{Y_{\varepsilon_j}\}$, where $j = 1, 2, 3$, $\varepsilon_1 = \{1, 12, 13, 123\}, \varepsilon_2 = \{2, 12, 23, 123\}, \varepsilon_3 = \{3, 13, 23, 123\}$.

5.5.3 Simulation Results and Comparisons

The following tables are the results of our simulation. To generate the point processes, we have used two distributions, one is the bivariate exponential distribution, the other is the bivariate Weibull distribution. To generate the damage vectors, we have used the trivariate exponential distribution for all the point processes. The parameters we used are explained as follows: For Table 5.4 to Table 5.7, p_1, p_2, p_3 are the parameters of the bivariate exponential distribution for two consecutive interarrival intervals, u_1, u_2, u_3 are the initial reserves, n is the sample size. For Table 5.8 to Table 5.11, p_1, p_2, p_3, β are the parameters of the bivariate Weibull distribution for two consecutive interarrival intervals, u_1, u_2, u_3 are the initial reserves, n is the sample size. For all the tables, the sample size n is 5000, the parameters of the damage vectors in case 1 and 2 are as follows:
Case 1: $\lambda_{123} = 0.5$, $\lambda_{23} = 1$, $\lambda_{13} = 1$, $\lambda_{12} = 1$, $\lambda_3 = 1$, $\lambda_2 = 1$, $\lambda_1 = 1$.
Case 2: $\lambda_{123} = 0.5$, $\lambda_{23} = 1$, $\lambda_{13} = 1$, $\lambda_{12} = 0.5$, $\lambda_3 = 1$, $\lambda_2 = 1.5$, $\lambda_1 = 1.5$.
Theorem 4.2.1 are vindicated by the simulation results.

Table 5.6: ψ_{sim} for the Bivariate Exponential Point Processes 1

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{sim}	0.0628	0.0458	0.0556	0.0516

$p_1 = 1.05, p_2 = 1.05, p_{12} = 0.05; u_1 = 1, u_2 = 1, u_3 = 1$

Table 5.7: ψ_{sim} for the Bivariate Exponential Point Processes 2

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{sim}	0.0540	0.0444	0.0608	0.0470

$p_1 = 1, p_2 = 1, p_{12} = 0.1; u_1 = 1, u_2 = 1, u_3 = 1$

Table 5.8: ψ_{and} and ψ_{or} for the Bivariate Exponential Point Processes 1

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{and}	0.0774	0.0672	0.0810	0.0694
ψ_{or}	0.5344	0.5364	0.5368	0.5486

$p_1 = 1.05, p_2 = 1.05, p_{12} = 0.05; u_1 = 3.5, u_2 = 3.5, u_3 = 3.5$

Table 5.9: ψ_{and} and ψ_{or} for the Bivariate Exponential Point Processes 2

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{and}	0.0764	0.0612	0.0630	0.0616
ψ_{or}	0.5154	0.5234	0.5150	0.5228

$p_1 = 1, p_2 = 1, p_{12} = 0.1; u_1 = 3.5, u_2 = 3.5, u_3 = 3.5$

Table 5.10: ψ_{sim} for the Bivariate Weibull Point Processes 1

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{sim}	0.0518	0.0438	0.0570	0.0446

$p_1 = 1.05, p_2 = 1.05, p_{12} = 0.05, \beta = 1.1; u_1 = 1, u_2 = 1, u_3 = 1$

Table 5.11: ψ_{sim} for the Bivariate Weibull Point Processes 2

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{sim}	0.0592	0.0496	0.0564	0.0458

$p_1 = 1, p_2 = 1, p_{12} = 0.1, \beta = 1.1; u_1 = 1, u_2 = 1, u_3 = 1$

Table 5.12: ψ_{and} and ψ_{or} for the Bivariate Weibull Point Processes 1

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{and}	0.0362	0.0256	0.0294	0.0268
ψ_{or}	0.3672	0.3706	0.3494	0.3596

$p_1 = 1.05, p_2 = 1.05, p_{12} = 0.05, \beta = 1.1; u_1 = 3.5, u_2 = 3.5, u_3 = 3.5$

Table 5.13: ψ_{and} and ψ_{or} for the Bivariate Weibull Point Processes 2

	Duplicate 1		Duplicate 2	
	Case 1	Case2	Case 1	Case 2
ψ_{and}	0.0370	0.0322	0.0318	0.0312
ψ_{or}	0.3920	0.3942	0.3790	0.3942

$p_1 = 1, p_2 = 1, p_{12} = 0.1, \beta = 1.1; u_1 = 3.5, u_2 = 3.5, u_3 = 3.5$

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Appendix A

BOUNDS.M

CODES

```
#-------------------------------------------------------#
# MATLAB codes for the computation of the upper and lower #
# bounds using ruin probabilities.                       #
# Codes written by Huajun Zhou, October 2005.            #
#                                                        #
# REQUIRES MATLAB VERSION 5.1 OR LATER.                  #
#-------------------------------------------------------#

# This function is to find Stochstic Bounds
# Input: u are initial m reserves
# Output: Phi1:Phi1(u1),Phi2:Phi2(u2),Phimin:Phimin(u(m)),
# Phimax: Phimax(u(1)),Phisum

function results=bounds2(u,lambda123,lambda23,lambda13,lambda12,
                lambda3,lambda2,lambda1,lambda0)

# lambda is the Poisson event arrival process of rate
# p1,p2,p3 are constants

lambda=1.6;
p=[3 3 3];
```

```
pmi=min(p);
pma=max(p);
umi=min(u);
uma=max(u);
Lambda=lambda123+lambda23+lambda13+lambda12+
       lambda3+lambda2+lambda1+lambda0;
A=[-Lambda+lambda23+lambda3+lambda2+lambda0 0 0 0 0 0 0;
   0 -Lambda+lambda13+lambda3+lambda1+lambda0 0 0 0 0 0;
   0 0 -Lambda+lambda12+lambda2+lambda1+lamdba0 0 0 0 0;
   lambda23+lambda2 lambda13+lambda1 0 -Lambda+lambda3+lambda0 0 0 0;
   lambda23+lambda3 0 lambda12+lambda1 0 -Lambda+lambda2+lambda0 0 0;
   0 lambda13+lambda3 lambda12+lambda2 0 0 -Lambda+lambda1+lambda0 0;
   lambda23 lambda13 lambda12 lambda3 lambda2 lambda1 -Lambda+lambda0];
Ainv=inv(A);

# E={12,2,1,0}, E1={12,1},E2={12,2}, A1 is the matrix of A(E-E1),
# A2 is the matrix of A(E-E2), A0 is -Lambda+lambda0

A1=[-Lambda+lambda23+lambda3+lambda2+lambda0 0 0 0;
   lambda23+lambda2 -Lambda+lambda3+lambda0 0 0;
   lambda23+lambda3 0 -Lambda+lambda2+lambda0 0;
   lambda23 lambda3 lambda2 -Lambda+lambda0];
A1inv=inv(A1); A2=[-Lambda+lambda13+lambda3+lambda1+lambda0 0 0 0;
   lambda13+lambda1 -Lambda+lambda3+lambda0 0 0;
   lambda13+lambda3 0 -Lambda+lambda1+lambda0 0;
   lambda13 lambda3 lambda1 -Lambda+lambda0];
A2inv=inv(A2); A3=[-Lambda+lambda12+lambda2+lambda1+lambda0 0 0 0;
   lambda12+lambda1 -Lambda+lambda2+lambda0 0 0;
   lambda12+lambda2 0 -Lambda+lambda1+lambda0 0;
   lambda12 lambda2 lambda1 -Lambda+lambda0];
A3inv=inv(A3);
A0=-Lambda+lambda0;
TT=[-Lambda+lambda23+lambda3+lambda2+lambda0 0 0 0 0 0 0;
   0 -Lambda+lambda13+lambda3+lambda1+lambda0 0 0 0 0 0;
   0 0 -Lambda+lambda12+lambda2+lambda1+lambda0 0 0 0 0;
```

```
   (lambda23+lambda2)/2 (lambda13+lambda1)/2 0
   (-Lambda+lambda3+lambda0)/2 0 0 0;
   (lambda23+lambda3)/2 0 (lambda12+lambda1)/2 0
   (-Lambda+lambda2+lambda0)/2 0 0;
   0 (lambda13+lambda3)/2 (lambda12+lambda2)/2 0 0
   (-Lambda+lambda1+lambda0)/2 0;
   lambda23/3 lambda13/3 lambda12/3 lambda3/3 lambda2/3 lambda1/3
   (-Lambda+lambda0)/3];
TTinv=inv(TT);
e=[1;1;1;1;1;1;1];
e1=[1;1;1;1];
e0=1;
alpha=[0 0 0 0 0 0 1];
alpha1=[0 0 0 1];
alpha2=[0 0 0 1];
alpha3=[0 0 0 1];
alpha0=1;
t1=-A1*e1;
t2=-A2*e1;
t3=-A3*e1;
tmin=-alpha0*e0;
tmax=-A*e;
tsum=-TT*e;
Phi1=-(lambda/p(1))*(alpha1*A1inv*expm((A1-(lambda/p(1))*
      (t1*alpha1*A1inv))*u(1))*e1);
Phi2=-(lambda/p(2))*(alpha2*A2inv*expm((A2-(lambda/p(2))*
      (t2*alpha2*A2inv))*u(2))*e1);
Phi3=-(lambda/p(3))*(alpha3*A3inv*expm((A3-(lambda/p(3))*
      (t3*alpha3*A3inv))*u(3))*e1);
Phimin=-lambda/(pma*A0)*exp((A0+lambda/pma)*uma);
Phimax=-(lambda/pmi)*(alpha*Ainv*expm((A-(lambda/pmi)*
        (tmax*alpha*Ainv))*umi)*e);
Phisum=-(lambda/sum(p))*(alpha*TTinv*expm((TT-(lambda/sum(p))*
        (tsum*alpha*TTinv))*sum(u))*e);
format long
```

```
results=[Phimin Phi1*Phi2*Phi3 Phisum Phisum
        1-(1-Phi1)*(1-Phi2)*(1-Phi3) Phimax];
```

Appendix B

TRI-EXP-SIM.SSC

CODES

```
#------------------------------------------------------------#
# S-PLUS codes for the simulation of the multivariate        #
# compound point process with drifts in which the shock      #
# arrival process is bivariate exponential distribution.     #
# This programming will compute the simultaneous ruin        #
# probability for the trivariate exponential damage vectors. #
# Codes written by Huajun Zhou, December 2005.               #
#                                                            #
# REQUIRES S-PLUS VERSION 4.5 OR LATER.                      #
#------------------------------------------------------------#
```

Exponential-simulation(Tri-variate Case)

```
# Given a vector, ssum is to output a vector of sums
# if a=[a1 a2 a3], then ssum=[ a1 a1+a2 a1+a2+a3]

ssum_function(a){
b_vector(length=length(a))
b[1]_a[1]
for (i in 2:length(a)){
    b[i]_a[i]+b[i-1]
```

```
}
b
}

# Given a vector, index0 is to output the index of
# first value of the vector whish is geater or equal to 0
# if all values are less than 0, then output 0
#
# Input: a vector;
# Output: an integer

indexgt_function(a){
    if (max(a) < 0) i_0
    else
        {i_1
         while(a[i] < 0 ){
         i_i+1
        }
    }
    i
}

# Simulation point process based on the previous point
# Bivariate Exponential Distribution
#
# Input: p1,p2,p12
# Output: x - the point processes

BiExp_function(p1,p2,p12){
total_100
t_0
i_1
x_vector()
Mt_p1+p2+p12
X_rexp(1,Mt)
```

```
T1_X U_runif(1,0,1)
lamdat_p2
while (t <= total) {
 if (lamdat/Mt > U)
     t_t+X
 else {x[i]_t+X;i_i+1; t_x[i-1]}

 X_rexp(1,Mt)
 U_runif(1,0,1)
 if (X > T1)
    lamdat_p2
 else lamdat_p12+p2

 T1_X
}
x[1:i-1]
}

now_proc.time()

# Input: lambda1,lambda2,lambda3,lambda12,lambda13,lambda23,lambda123,
#         u1,u2,u3
# Output: counter1
# Counter1 is for PHI_sim

counter1_c(0,0)
N_5000
for (j in 1:N) {
x_BiExp(1.0,1.0,0.1)
n_length(x)
point_c(1:n)

# Below are the inputs

lambda1_c(1,1.5)
```

```
lambda2_c(1,1.5)
lambda3_c(1,1)
lambda12_c(1,0.5)
lambda13_c(1,1)
lambda23_c(1,1)
lambda123_c(0.5,0.5)
u1_1.0 u2_1.0 u3_1.0
Y1_rexp(n,lambda1[1])
Y2_rexp(n,lambda2[1])
Y3_rexp(n,lambda3[1])
Y12_rexp(n,lambda12[1])
Y13_rexp(n,lambda13[1])
Y23_rexp(n,lambda23[1])
Y123_rexp(n,lambda123[1])
X1_pmin(Y1,Y12,Y13,Y123)
X2_pmin(Y2,Y12,Y23,Y123)
X3_pmin(Y3,Y13,Y23,Y123)
S1_ssum(X1)-x*point/100-u1
S2_ssum(X2)-x*point/100-u2
S3_ssum(X3)-x*point/100-u3

if (indexgt(S1) > 0 && indexgt(S1) == indexgt(S2) && indexgt(S2) ==
    indexgt(S3))
    counter1[1]_counter1[1]+1
else
    counter1[1]_counter1[1]

#*******************************

Z1_rexp(n,lambda1[2])
Z2_rexp(n,lambda2[2])
Z3_rexp(n,lambda3[2])
Z12_rexp(n,lambda12[2])
Z13_rexp(n,lambda13[2])
Z23_rexp(n,lambda23[2])
```

```
Z123_rexp(n,lambda123[2])
W1_pmin(Z1,Z12,Z13,Z123)
W2_pmin(Z2,Z12,Z23,Z123)
W3_pmin(Z3,Z13,Z23,Z123)
V1_ssum(W1)-x*point/100-u1
V2_ssum(W2)-x*point/100-u2
V3_ssum(W3)-x*point/100-u3
if (indexgt(V1) > 0 && indexgt(V1) == indexgt(V2) && indexgt(V2)
    == indexgt(V3))
    counter1[2]_counter1[2]+1
else
    counter1[2]_counter1[2]

}
SS1_(counter1*(1-(counter1/N))^2+(N-counter1)*(counter1/N)^2)/(N-1)
counter1/N
SS1
proc.time()-now
```

Appendix C

TRI-EXP-AND-OR.SSC

CODES

```
#--------------------------------------------------------------#
# S-PLUS codes for the simulation of the multivariate          #
# compound point process with drifts in which the shock        #
# arrival process is bivariate exponential distribution.       #
# This programming will compute the and/or ruin                #
# probability for the trivariate exponential damage vectors. #
# Codes written by Huajun Zhou, December 2005.                 #
#                                                              #
# REQUIRES S-PLUS VERSION 4.5 OR LATER.                         #
#--------------------------------------------------------------#

# Exponential-simulation(Tri-variate Case)

# Given a vector, ssum is to output a vector of sums
# if a=[a1 a2 a3], then ssum=[ a1 a1+a2 a1+a2+a3]

ssum_function(a){
b_vector(length=length(a))
b[1]_a[1]
for (i in 2:length(a)) {
```

```
    b[i]_a[i]+b[i-1]
}
b
}

# Given a vector, index0 is to output the index of
# first value of the vector whish is geater or equal to 0
# if all values are less than 0, then output 0
# Input: a vector
# Output: an integer

indexgt_function(a){
    if (max(a) < 0) i_0
    else
    {i_1
    while(a[i] < 0 ){
        i_i+1
        }
        }
    i
}

# Simulation point process based on the previous point
# Bivariate Exponential Distribution
# Input: p1,p2,p12
# Output: x: are the point processes

BiExp_function(p1,p2,p12){
total_100
t_0
i_1
x_vector()
Mt_p1+p2+p12
X_rexp(1,Mt)
T1_X
```

```
U_runif(1,0,1)
lamdat_p2
while (t <= total) {
    if (lamdat/Mt > U) t_t+X
    else {x[i]_t+X;i_i+1; t_x[i-1]}
    X_rexp(1,Mt)
    U_runif(1,0,1)
    if (X > T1) lamdat_p2
    else lamdat_p12+p2
    T1_X
}
x[1:i-1]
}


now_proc.time()

# Input: lamda1,lamda2,lamda3,lamda12,lamda13,lamda23,lamda123,
#        u1,u2,u3
# Output: counter1,counter2,counter3,
# Counter2 is for PHI_and, counter3 is for PHI_or

counter2_c(0,0)
counter3_c(0,0)
N_5000
for (j in 1:N) {
x_BiExp(1.05,1.05,0.05)
n_length(x)
point_c(1:n)

# Below are the inputs

lamda1_c(1,1.5)
lamda2_c(1,1.5)
lamda3_c(1,1)
```

```
lamda12_c(1,0.5)
lamda13_c(1,1)
lamda23_c(1,1)
lamda123_c(0.5,0.5)
u1_3.5
u2_3.5
u3_3.5
Y1_rexp(n,lamda1[1])
Y2_rexp(n,lamda2[1])
Y3_rexp(n,lamda3[1])
Y12_rexp(n,lamda12[1])
Y13_rexp(n,lamda13[1])
Y23_rexp(n,lamda23[1])
Y123_rexp(n,lamda123[1])
X1_pmin(Y1,Y12,Y13,Y123)
X2_pmin(Y2,Y12,Y23,Y123)
X3_pmin(Y3,Y13,Y23,Y123)
S1_ssum(X1)-x*point/100-u1
S2_ssum(X2)-x*point/100-u2
S3_ssum(X3)-x*point/100-u3
if (max(S1) >= 0 && max(S2) >= 0 && max(S3) >= 0 )
    counter2[1]_counter2[1] + 1
else
    counter2[1]_counter2[1]
if (max(S1,S2) >= 0 || max(S1,S3) >= 0)
    counter3[1]_counter3[1] + 1
else
    counter3[1]_counter3[1]

#*******************************

Z1_rexp(n,lamda1[2])
Z2_rexp(n,lamda2[2])
Z3_rexp(n,lamda3[2])
Z12_rexp(n,lamda12[2])
```

```
Z13_rexp(n,lamda13[2])
Z23_rexp(n,lamda23[2])
Z123_rexp(n,lamda123[2])
W1_pmin(Z1,Z12,Z13,Z123)
W2_pmin(Z2,Z12,Z23,Z123)
W3_pmin(Z3,Z13,Z23,Z123)
V1_ssum(W1)-x*point/100-u1
V2_ssum(W2)-x*point/100-u2
V3_ssum(W3)-x*point/100-u3
if (max(V1) >= 0 && max(V2) >= 0 && max(V3) >= 0 )
    counter2[2]_counter2[2] + 1
else
    counter2[2]_counter2[2]
if (max(V1,V2) >= 0 || max(V1,V3) >= 0)
    counter3[2]_counter3[2] + 1
else
    counter3[2]_counter3[2]

}
SS2_(counter2*(1-(counter2/N))^2+(N-counter2)*(counter2/N)^2)/(N-1)
SS3_(counter3*(1-(counter3/N))^2+(N-counter3)*(counter3/N)^2)/(N-1)
counter2/N
counter3/N
SS2
SS3
proc.time()-now
```

Appendix D

TRI-WEI-SIM.SSC

CODES

```
#------------------------------------------------------------#
# S-PLUS codes for the simulation of the multivariate        #
# compound point process with drifts in which the shock      #
# arrival process is bivariate weibull distribution.         #
# This programming will compute the simultaneous ruin        #
# probability for the trivariate exponential damage vectors. #
# Codes written by Huajun Zhou, December 2005.               #
#                                                            #
# REQUIRES S-PLUS VERSION 4.5 OR LATER.                      #
#------------------------------------------------------------#

# Weibull-simulation-sim(Tri-variate Case)
# Given a vector, ssum is to output a vector of sums
# if a=[a1 a2 a3], then ssum=[ a1 a1+a2 a1+a2+a3]

ssum_function(a){
b_vector(length=length(a))
b[1]_a[1]
for (i in 2:length(a)) {
    b[i]_a[i]+b[i-1]
}
```

```
b
}

# Given a vector, index0 is to output the index of
# first value of the vector whish is geater or equal to 0
# if all values are less than 0, then output 0
# Input: a vector
# output: an integer

indexgt_function(a){
    if (max(a) < 0) i_0
        else
    {i_1
    while(a[i] < 0 ){
        i_i+1
        }
        }
    i
}

# Simulation point process based on the previous point
# Bivariate Weibull Distribution
# Input: p1,p2,p12,beta
# Output: x: are the point processes

BiWei_function(p1,p2,p12,beta){
total_100
t_0
i_1
x_vector()
Mt_beta*(p2+p12)*100^(beta-1)
X_rexp(1,Mt)
T1_X
U_runif(1,0,1)
lamdat_beta*p2*X^(beta-1)
```

```
while (t <= total) {
if (lamdat/Mt > U) t_t+X
else {x[i]_t+X;i_i+1; t_x[i-1]}
X_rexp(1,Mt)
U_runif(1,0,1)
if (X > T1) lamdat_beta*p2*X^(beta-1)
else lamdat_beta*(p2+p12)*X^(beta-1)
T1_X
}
x[1:i-1]
}

now_proc.time()

# Input: lamda1,lamda2,lamda3,lamda12,lamda13,lamda23,lamda123,
#        u1,u2,u3
# Output: counter1
# Counter1 is for PHI_sim, counter2 is for PHI_and,
#           counter3 is for PHI_or

counter1_c(0,0)
N_5000
for (j in 1:N) {
x_BiWei(1.05,1.05,0.05,1.1)
n_length(x)
point_c(1:n)

# Below are the inputs

lamda1_c(1,1.5)
lamda2_c(1,1.5)
lamda3_c(1,1)
lamda12_c(1,0.5)
lamda13_c(1,1)
lamda23_c(1,1)
```

```
lamda123_c(0.5,0.5)
u1_1
u2_1
u3_1
Y1_rexp(n,lamda1[1])
Y2_rexp(n,lamda2[1])
Y3_rexp(n,lamda3[1])
Y12_rexp(n,lamda12[1])
Y13_rexp(n,lamda13[1])
Y23_rexp(n,lamda23[1])
Y123_rexp(n,lamda123[1])
X1_pmin(Y1,Y12,Y13,Y123)
X2_pmin(Y2,Y12,Y23,Y123)
X3_pmin(Y3,Y13,Y23,Y123)
S1_ssum(X1)-x*point/100-u1
S2_ssum(X2)-x*point/100-u2
S3_ssum(X3)-x*point/100-u3
if (indexgt(S1) > 0 && indexgt(S1) == indexgt(S2) &&
    indexgt(S2) == indexgt(S3))
    counter1[1]_counter1[1]+1
else
    counter1[1]_counter1[1]

#*******************************

Z1_rexp(n,lamda1[2])
Z2_rexp(n,lamda2[2])
Z3_rexp(n,lamda3[2])
Z12_rexp(n,lamda12[2])
Z13_rexp(n,lamda13[2])
Z23_rexp(n,lamda23[2])
Z123_rexp(n,lamda123[2])
W1_pmin(Z1,Z12,Z13,Z123)
W2_pmin(Z2,Z12,Z23,Z123)
W3_pmin(Z3,Z13,Z23,Z123)
```

```
V1_ssum(W1)-x*point/100-u1
V2_ssum(W2)-x*point/100-u2
V3_ssum(W3)-x*point/100-u3
if (indexgt(V1) > 0 && indexgt(V1) == indexgt(V2)
    && indexgt(V2) == indexgt(V3))
    counter1[2]_counter1[2]+1
else
    counter1[2]_counter1[2]
}
SS1_(counter1*(1-(counter1/N))^2+(N-counter1)*(counter1/N)^2)/(N-1)
counter1/N
SS1
proc.time()-now
```

Appendix E

TRI-WEI-AND-OR.SSC

CODES

```
#-------------------------------------------------------------#
# S-PLUS codes for the simulation of the multivariate         #
# compound point process with drifts in which the shock       #
# arrival process is bivariate weibull distribution.          #
# This programming will compute the and/or ruin               #
# probability for the trivariate exponential damage vectors. #
# Codes written by Huajun Zhou, December 2005.                #
#                                                             #
# REQUIRES S-PLUS VERSION 4.5 OR LATER.                       #
#-------------------------------------------------------------#

# Weibull-simulation-and-or(Tri-variate Case)
# Given a vector, ssum is to output a vector of sums
# if a=[a1 a2 a3], then ssum=[ a1 a1+a2 a1+a2+a3]

ssum_function(a){
b_vector(length=length(a))
b[1]_a[1]
for (i in 2:length(a)) {
    b[i]_a[i]+b[i-1]
```

```
}
b
}

# Given a vector, index0 is to output the index of
# first value of the vector whish is geater or equal to 0
# if all values are less than 0, then output 0
# Input: a vector
# Output: an integer

indexgt_function(a){
    if (max(a) < 0) i_0
        else
    {i_1
    while(a[i] < 0 ){
        i_i+1
        }
        }
    i
}

# Simulation point process based on the previous point
# Bivariate Weibull Distribution
# Input: p1,p2,p12,beta
# Output: x: are the point processes

BiWei_function(p1,p2,p12,beta){
total_100
t_0
i_1
x_vector()
Mt_beta*(p2+p12)*100^(beta-1)
X_rexp(1,Mt)
T1_X
U_runif(1,0,1)
```

```
lamdat_beta*p2*X^(beta-1)
while (t <= total) {
if (lamdat/Mt > U) t_t+X
        else {x[i]_t+X;i_i+1; t_x[i-1]}
    X_rexp(1,Mt)
    U_runif(1,0,1)
if (X > T1) lamdat_beta*p2*X^(beta-1)
    else lamdat_beta*(p2+p12)*X^(beta-1)
T1_X
}
x[1:i-1]
}

now_proc.time()

# Input: lamda1,lamda2,lamda3,lamda12,lamda13,lamda23,lamda123,
#        u1,u2,u3
# Output: counter1,counter2,counter3,
# Counter1 is for PHI_sim,counter2 is for PHI_and,
#          counter3 is for PHI_or

counter2_c(0,0)
counter3_c(0,0)
N_5000
for (j in 1:N) {
x_BiWei(1,1,0.1,1.1)
n_length(x)
point_c(1:n)
# Below are the inputs
lamda1_c(1,1.5)
lamda2_c(1,1.5)
lamda3_c(1,1)
lamda12_c(1,0.5)
lamda13_c(1,1)
lamda23_c(1,1)
```

```
lamda123_c(0.5,0.5)
u1_3.5
u2_3.5
u3_3.5
Y1_rexp(n,lamda1[1])
Y2_rexp(n,lamda2[1])
Y3_rexp(n,lamda3[1])
Y12_rexp(n,lamda12[1])
Y13_rexp(n,lamda13[1])
Y23_rexp(n,lamda23[1])
Y123_rexp(n,lamda123[1])
X1_pmin(Y1,Y12,Y13,Y123)
X2_pmin(Y2,Y12,Y23,Y123)
X3_pmin(Y3,Y13,Y23,Y123)
S1_ssum(X1)-x*point/100-u1
S2_ssum(X2)-x*point/100-u2
S3_ssum(X3)-x*point/100-u3
if (max(S1) >= 0 && max(S2) >= 0 && max(S3) >= 0 )
    counter2[1]_counter2[1] + 1
else
    counter2[1]_counter2[1]
if (max(S1,S2) >= 0 || max(S1,S3) >= 0)
    counter3[1]_counter3[1] + 1
else
    counter3[1]_counter3[1]

#********************************

Z1_rexp(n,lamda1[2])
Z2_rexp(n,lamda2[2])
Z3_rexp(n,lamda3[2])
Z12_rexp(n,lamda12[2])
Z13_rexp(n,lamda13[2])
Z23_rexp(n,lamda23[2])
Z123_rexp(n,lamda123[2])
```

```
W1_pmin(Z1,Z12,Z13,Z123)
W2_pmin(Z2,Z12,Z23,Z123)
W3_pmin(Z3,Z13,Z23,Z123)
V1_ssum(W1)-x*point/100-u1
V2_ssum(W2)-x*point/100-u2
V3_ssum(W3)-x*point/100-u3
if (max(V1) >= 0 && max(V2) >= 0 && max(V3) >= 0 )
    counter2[2]_counter2[2] + 1
else
    counter2[2]_counter2[2]
if (max(V1,V2) >= 0 || max(V1,V3) >= 0)
    counter3[2]_counter3[2] + 1
else
    counter3[2]_counter3[2]
}
SS2_(counter2*(1-(counter2/N))^2+(N-counter2)*(counter2/N)^2)/(N-1)
SS3_(counter3*(1-(counter3/N))^2+(N-counter3)*(counter3/N)^2)/(N-1)
counter2/N
counter3/N
SS2
SS3
proc.time()-now
```