COMBINATORIAL PROPERTIES OF NONNEGATIVE AND EVENTUALLY NONNEGATIVE MATRICES

By

DEANNE MORRIS

A dissertation submitted in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

WASHINGTON STATE UNIVERSITY
Department of Mathematics

AUGUST 2008

I certify that I have read this thesis and certify that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Chair

COMBINATORIAL PROPERTIES OF NONNEGATIVE AND EVENTUALLY NONNEGATIVE MATRICES

Abstract

by DEANNE MORRIS, Ph.D.
Washington State University
Department of Mathematics
AUGUST 2008

Chair: Judith J. McDonald

Nonnegative and eventually nonnegative matrices are useful in many areas of mathematics and have been widely studied. Perhaps the most famous results pertaining to nonnegative matrices are due to Perron and Frobenius and have prompted much further research. In this thesis, we investigate the combinatorial structure of nonnegative and eventually nonnegative matrices as it relates to the (peripheral) Jordan form of the matrices. Information obtained from the level form of a nonnegative matrix is used to gain insight into the Jordan form of the matrix, and vice-versa.

In this paper, we give necessary and sufficient conditions for a set of Jordan blocks to correspond to the peripheral spectrum of a nonnegative matrix. For each eigenvalue, λ, the λ-level characteristic (with respect to the spectral radius) is defined. The necessary and sufficient conditions include a requirement that the λ-level characteristic is majorized by the λ-height characteristic. An algorithm which has been implemented in MATLAB is given to determine when a multiset of Jordan blocks corresponds to the peripheral spectrum of a nonnegative matrix.

We also consider the Jordan form of an eventually nonnegative matrix. It is known that the necessary and sufficient conditions for a multiset of Jordan blocks to correspond to the peripheral Jordan form of an eventually nonnegative matrix coincide exactly with the necessary and sufficient conditions for a multiset of Jordan blocks to correspond to the peripheral Jordan form of a nonnegative matrix. We take a closer look at the Jordan blocks associated with eigenvalues of smaller magnitude. Multiple sufficient conditions on the Jordan form of an (eventually) nonnegative matrix are given. We also offer necessary and sufficient conditions on the Jordan form of an eventually nonnegative matrix.

Acknowledgment

There are so many people who have helped me throughout my time here at Washington State University. I would like to thank my wonderful adviser, Judi McDonald, who has spent so much of her own time in helping me to become a research mathematician. Her guidance, both personal and professional, has been invaluable to me. Over the years, she has taught me a great deal, and has been infinitely patient and kind. She has passed on to me an excitement for the work that we do and has increased my confidence as a mathematician and teacher. I am grateful for many other professors who have taught me a great deal and have been supportive, especially those on my committee: Alan Genz, Matt Hudelson, and Michael Tsatsomeros. I have had the pleasure of taking a course from each of these mathematicians and know first hand that they are not only talented researchers, but exceptional educators as well.

I must also thank my loving family without whom none of this would be possible. My husband Bryan and son Peyton have been constant sources of joy and motivation for me. My parents, Bev and Larry Lamkin, have encouraged me my whole life and have always been there for me. I could never thank them enough for all that they have done for me. I can always count on my sister, Lisa, for a refreshing good laugh and reminder of the fact that family is what's really important. Last, but not least, I am thankful for the friends I have made here; Amy, Jason, TJ, Carly, Liz, Olga, and Jongsam (just to name a few) have been supportive, both personally and professionally. I could always count on Amy for good conversation, plain old fun, and encouragement on those days when life seemed overwhelming.

List of Symbols

Suppose A is a matrix and $\mathcal{J}$ is a multiset of matrices.

$\langle n \rangle = \{1, 2, \ldots, n\}$

$\sigma(A) = \{\lambda \mid \lambda \text{ is an eigenvalue of } A\ \}$: spectrum of A, a multiset

$\rho(A) = \max_{\lambda \in \sigma(A)}\{|\lambda|\}$: spectral radius of A

$\pi(A) = \{\lambda \in \sigma(A) \mid |\lambda| = \rho(A)\}$: peripheral spectrum of A, a multiset

$P(A) = \{|\lambda| \mid \lambda \in \sigma(A)\}$, a set

$\mathrm{index}_\lambda(A) =$ number of times λ occurs as a root of the minimal polynomial of A

$\mathbb{Z}_q = \{1, e^{\frac{2\pi i}{q}}, e^{\frac{4\pi i}{q}}, \ldots, e^{\frac{2(q-1)\pi i}{q}}\}$: complete set of q^{th} roots of unity

$\eta(A) = (\eta_1(A), \eta_2(A), \ldots, \eta_m(A))$ where $\eta_i(A) = \mathrm{nullity}(A^i) - \mathrm{nullity}(A^{i-1})$: height characteristic of A

$\eta_\lambda(A) = \eta(A - \lambda I)$: λ-height characteristic of A

$G(A)$: the (directed) graph of A

$\mathcal{R}(A)$: the reduced graph of A

$\nu(A) = (\nu_1(A), \ldots, \nu_m(A))$ where $\nu_i(A)$ is the number of singular vertices with level i in $\mathcal{R}(A)$: level characteristic of A

$\nu_{\lambda,\rho}(A) = (\nu_1(A), \ldots, \nu_m(A))$ where $\nu_i(A)$ is the number of times λ occurs as an eigenvalue of level i of A : λ-level characteristic of A (with respect to $\rho = \rho(A)$)

$J_n(\lambda)$: the $n \times n$ Jordan block with eigenvalue λ

$\mathcal{J}(\lambda) = \{J \in \mathcal{J} \mid \lambda \in \sigma(J)\}$

$\mathcal{J}^\rho = \{J \in \mathcal{J} \mid \text{ if } \lambda \in \sigma(J), \text{ then } |\lambda| = \rho\}$

$\mathcal{J}(A)$: the multiset of Jordan blocks corresponding to the Jordan form of A

$\oplus\mathcal{J}$: the matrix obtained by taking the direct sum of elements of $\mathcal{J}$

$\sigma(\mathcal{J}) = \sigma(\oplus\mathcal{J}), \qquad \rho(\mathcal{J}) = \rho(\oplus\mathcal{J}), \qquad \pi(\mathcal{J}) = \pi(\oplus\mathcal{J}),$

$\text{index}_\lambda(\mathcal{J}) = \text{index}_\lambda(\oplus\mathcal{J}) =$ size of largest Jordan block in $\mathcal{J}$ with eigenvalue λ

List of Figures

1 Reduced Graph of A 10

2 Reduced Graph of A 14

3 Search tree for $m = 3$ and $\sigma(\mathcal{J}) = \mathbb{Z}_{R_1} \bigcup \mathbb{Z}_{R_2} \bigcup \mathbb{Z}_{R_3} \bigcup \mathbb{Z}_{R_4}$ 27

4 Pruned search tree for $m = 3$ and $\sigma(\mathcal{J}) = \mathbb{Z}_{R_1} \bigcup \mathbb{Z}_{R_2} \bigcup \mathbb{Z}_{R_3} \bigcup \mathbb{Z}_{R_4}$ 27

5 The reduced graph of A from which we determine the split-level partition of A 35

6 The reduced graph of an upper level of A. 36

Contents

1 Introduction and Standard Definitions **1**

1.1 Introduction 1

1.2 Standard Definitions and Notation 3

1.3 Historical Results 8

2 The Peripheral Jordan Form of a Nonnegative Matrix **15**

2.1 Necessary and Sufficient Conditions on $\mathcal{J}$ 15

2.2 An Algorithmic Approach 25

3 The Jordan Form of an Eventually Nonnegative Matrix **30**

3.1 Jordan Forms and Frobenius Normal Forms of Powers of A 31

3.2 Component Level Sets 34

3.3 Sufficient Conditions on $\mathcal{J}$ 40

3.4 Necessary and Sufficient Conditions on $\mathcal{J}$ 48

4 Concluding Remarks **65**

A MATLAB Code for level_periph **67**

B MATLAB Code for J2NN **79**

Bibliography **86**

Chapter 1

Introduction and Standard Definitions

1.1 Introduction

The Perron-Frobenius Theorem is an important result which has prompted much research pertaining to spectral properties of nonnegative matrices. Many combinatorial properties of the spectrum of a nonnegative matrix, and its generalized eigenspace, are known (for example, see [1], [5], and [13]). Many authors have contributed to this knowledge base, and there are many interesting papers on this topic.

While introducing the idea of level sets associated with a nonnegative matrix, Richman and Schneider give results pertaining to the singular graph and Weyr characteristic of an M-matrix in [11] as does Rothblum in [12]. Hershkowitz and Schneider give necessary and sufficient conditions on the relation between the height and level characteristics corresponding to the spectral radius of a nonnegative matrix in [6]. For an overview of these results and many more on this topic, see the survey papers [5] and [13]. In particular, it has been shown that, if ν and η are two sequences of nonnegative integers, then there exists a nonnegative matrix A with height characteristic η, corresponding to the spectral radius, and level characteristic ν, corresponding to the spectral radius (with entries rewritten in decreasing order), if and only if ν is majorized by η.

Looking at extensions of Perron-Frobenius theory from a cone theoretic perspective, Tam and Schneider obtain a necessary and sufficient condition on the peripheral spectrum of a matrix for which there is a proper cone that the matrix leaves invariant, with the core of the matrix with respect to this cone being simplicial [15]. Tam noted that the condition is also necessary for a nonnegative matrix and called for a matrix theoretic proof of this necessity, with the question being formally posed in [14]. In [9], McDonald extends the necessary condition and offers a necessary and sufficient condition (the extended Tam-Schneider condition) for a multiset of Jordan blocks to correspond to the peripheral spectrum of a nonnegative matrix A.

In Chapter 2, we offer necessary and sufficient conditions which are equivalent to the extended Tam-Schneider condition, but which are more concise and point out the relation between level sets and the height characteristic of A corresponding to the spectral radius. For each eigenvalue λ in the peripheral spectrum of a nonnegative matrix A, we define the $\lambda-$level characteristic (with respect to the spectral radius) and show that the $\lambda-$level characteristic is majorized by the $\lambda-$height characteristic. This property and the requirement that the peripheral spectrum associated with each level must be a union of complete sets of roots of unity (multiplied by the spectral radius of A) provide necessary and sufficient conditions on the peripheral spectrum of a nonnegative matrix. This result is formally stated as Theorem 2.9. An algorithm is presented which determines whether or not there exists a nonnegative matrix with peripheral spectrum corresponding to a given multiset $\mathcal{J}$ of Jordan blocks. The algorithm is based on the conditions given in Theorem 2.9 and has been implemented in MATLAB.

In Chapter 3, we are interested in the entire spectrum (as opposed to just the peripheral spectrum) and present results related to the question of when a multiset of Jordan blocks corresponds to the spectrum of an eventually nonnegative matrix. It is known that the peripheral spectrum of an eventually nonnegative matrix must satisfy the same conditions given for the peripheral spectrum of a nonnegative matrix, so the difficulty lies in analyzing the Jordan form of the eigenvalues with smaller modulus. We will see that the Jordan form of an eventually nonnegative matrix satisfies many beautiful, yet complicated properties. Beginning with the idea of split-level partitions, we will define component level partitions of eventually nonnegative matrices and discuss combinatorial properties of their Jordan forms in this context.

1.2 Standard Definitions and Notation

Let $A \in \mathbb{C}^{n \times n}$.

For any $c \in \mathbb{C}$, $\bar{c}$ represents the complex conjugate of c. For a matrix A, $\bar{A}$ represents the matrix formed from A by conjugating each entry.

We will write $\langle n \rangle$ for $\{1, \ldots, n\}$.

$\mathbb{Z}_q$ is defined to be the set $\{1, e^{\frac{2\pi i}{q}}, e^{\frac{4\pi i}{q}}, \ldots, e^{\frac{2(q-1)\pi i}{q}}\}$ and is referred to as a *complete set of roots of unity* or the q^{th} *roots of unity.*

The multiset

$$\sigma(A) = \{\lambda \mid \lambda \text{ is an eigenvalue of } A \},$$

where each eigenvalue is listed the number of times it occurs as a root of the characteristic polynomial of A, is referred to as the *spectrum* of A.

$$\rho(A) = \max_{\lambda \in \sigma(A)} \{|\lambda|\}$$

is the *spectral radius* of A. The multiset

$$\pi(A) = \{\lambda \in \sigma(A) \mid |\lambda| = \rho(A)\}$$

is referred to as the *peripheral spectrum* of A. We let $\text{mult}_\lambda(A)$ denote the degree of λ as a root of the characteristic polynomial of A, and $\text{index}_\lambda(A)$ denote the degree of λ as a root of the minimal polynomial of A. Let $m = \text{index}_0(A)$, and for each $i \in \langle m \rangle$, set $\eta_i(A) = \text{nullity}(A^i) - \text{nullity}(A^{i-1})$. The sequence $\eta(A) = (\eta_1(A), \eta_2(A), \ldots, \eta_m(A))$ is referred to as the *height* or *Weyr characteristic* of A. The height characteristic of $A - \lambda I$ is referred to as the *λ-height characteristic of* A and is denoted by $\eta_\lambda(A)$. If $\mathcal{J}$ is a multiset of matrices, we let J be the direct sum of the matrices from $\mathcal{J}$ and define the λ-height characteristic of $\mathcal{J}$ to be the λ-height characteristic of J.

Let $\eta = (\eta_1, \eta_2, \ldots, \eta_t)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_t)$ be two sequences of nonnegative integers (append zeros if necessary to the end of the shorter sequence so that they are of the same length). Then ν is *majorized* by η if $\sum_{i=1}^{j} \nu_i \leq \sum_{i=1}^{j} \eta_i$, for all $1 \leq j \leq t$, and $\sum_{i=1}^{t} \nu_i = \sum_{i=1}^{t} \eta_i$. We write $\nu \preceq \eta$. We denote by $\hat{\nu}$ the sequence ν reordered in decreasing order.

We write $J_j(\lambda)$ to represent the $j \times j$ matrix whose diagonal elements are λ, whose first subdiagonal elements are 1, and all other elements are zero. We will refer to such a matrix as a *Jordan block (with eigenvalue λ).*

We say a collection of Jordan blocks $\mathcal{J}$ is self–conjugate if, whenever $J_j(\lambda) \in \mathcal{J}$ and λ is not real, we have $J_j(\bar{\lambda}) \in \mathcal{J}$ and the two blocks occur the same number of times.

We say that a collection of Jordan blocks $\mathcal{J}$ *corresponds* to the Jordan form of A provided the Jordan form of A is the direct sum of the elements in $\mathcal{J}$.

Let $\mathcal{J}$ be a collection of Jordan blocks and let J be the direct sum of the elements in $\mathcal{J}$. We define the *spectrum of* $\mathcal{J}$ by $\sigma(\mathcal{J}) = \sigma(J)$, the *spectral radius of* $\mathcal{J}$ by $\rho(\mathcal{J}) = \rho(J)$, the *peripheral spectrum of* $\mathcal{J}$ by $\pi(\mathcal{J}) = \pi(J)$, and the *$\lambda$-height characteristic of* $\mathcal{J}$ by $\eta_\lambda(\mathcal{J}) = \eta_\lambda(J)$. We denote the direct sum of all blocks in $\mathcal{J}$ by $\oplus\mathcal{J}$. We let $P(\mathcal{J})$

be the set (not multiset) $\{|\lambda| \mid \lambda \in \sigma(\mathcal{J})\}$ and $\mathcal{J}^{\rho_j}$ the multiset of Jordan blocks from $\mathcal{J}$ whose eigenvalues have magnitude ρ_j. For $\lambda \in \sigma(\mathcal{J})$, we let $\mathcal{J}(\lambda)$ be the multiset of Jordan blocks from $\mathcal{J}$ with eigenvalue λ. If A is a square matrix, we denote by $\mathcal{J}(A)$, the multiset of Jordan blocks corresponding to the Jordan form of A.

A multiset of Jordan blocks $\mathcal{J}$ is said to be a *Frobenius multiset* if for some positive integer h the following conditions are satisfied:

(i) there is exactly one block in $\mathcal{J}$ with eigenvalue $\rho(\mathcal{J})$ and this block is 1×1,

(i) $\pi(\mathcal{J}) = \rho(\mathcal{J})\mathbb{Z}_h$, and

(ii) $e^{\frac{2\pi i}{h}}\mathcal{J} = \mathcal{J}$.

We note that our definition differs from that of [16] and [10] in that we do not require $\rho(\mathcal{J}) > 0$ so we do consider $\{[0]\}$ to be a Frobenius multiset.

A matrix $A \in \mathbb{R}^{n \times n}$ is called:

positive $(A \gg 0)$ if $a_{ij} > 0$, for all $i, j \in \langle n \rangle$;

semipositive $(A > 0)$ if $a_{ij} \geq 0$, for all $i, j \in \langle n \rangle$ and $A \neq 0$; and

nonnegative $(A \geq 0)$ if $a_{ij} \geq 0$, for all $i, j \in \langle n \rangle$.

Let $\Gamma = (V, E)$ be a (directed) graph, where V is a finite vertex set and $E \subseteq V \times V$ is an edge set. A *path* from j to l in Γ is a sequence of vertices $j = r_1, r_2, ..., r_t = l$, with $(r_i, r_{i+1}) \in E$, for $i = 1, ..., t-1$. A path for which the vertices are pairwise distinct is called a *simple path*. The empty path will be considered to be a simple path linking every vertex to itself.

We define the *graph* of A by $G(A) = (V, E)$, where $V = \langle n \rangle$ and $E = \{(i, j) \mid a_{ij} \neq 0\}$.

Let $\Gamma = (V, E)$ be a graph. If there is a path from a vertex j to a vertex l in Γ, we say that j has *access* to l. If j has access to l and l has access to j, we say j and

l *communicate*. The communication relation is an equivalence relation, hence we may partition V into equivalence classes, which we will refer to as the *(irreducible) classes* of Γ. Note that this characterization of irreducibility requires that the 1×1 zero matrix be irreducible.

Let $K, L \subseteq \langle n \rangle$. We will write A_{KL} to represent the submatrix of A whose rows are indexed from K and whose columns are indexed from L. If $\kappa = (K_1, K_2, \ldots, K_k)$ is an ordered partition of $\langle n \rangle$, we write

$$A_\kappa = \begin{bmatrix} A_{K_1K_1} & A_{K_1K_2} & \ldots & A_{K_1K_k} \\ A_{K_2K_1} & A_{K_2K_2} & \ldots & A_{K_2K_k} \\ \vdots & \vdots & & \vdots \\ A_{K_kK_1} & A_{K_kK_2} & \ldots & A_{K_kK_k} \end{bmatrix}.$$

We say A_κ is *block lower triangular* if $A_{K_iK_j} = 0$ whenever $i < j$. We refer to the blocks $A_{K_iK_j}$ as *subdiagonal* blocks whenever $i > j$. Given a matrix A, it is well known that there is an ordered partition $\kappa = (K_1, K_2, \ldots, K_k)$ of $\langle n \rangle$ so that each K_i corresponds to a class of $G(A)$ and A_κ is block lower triangular. We say that A_κ is the *Frobenius normal form* of A. A class K_j is said to be *singular* if $A_{K_jK_j}$ is singular, and *nonsingular* otherwise. If A is an eventually nonnegative matrix, a class K_j of A is said to be *basic* if $\rho(A_{K_jK_j}) = \rho(A)$, and *nonbasic* otherwise.

We define the *reduced graph of* A by $\mathcal{R}(A) = (V, E)$ where $V = \{\ K \mid K$ is a class of $A\ \}$, and $E = \{\ (K, L) \mid$ there is edge from a vertex $j \in K$ to a vertex $l \in L$ in $G(A)\ \}$.

The *singular length* of a simple path in $\mathcal{R}(A)$ is the sum of the indexes of zero of each of the singular vertices it contains. The *level* of a vertex K is the maximum singular length over all the simple paths in $\mathcal{R}(A)$ which terminate at K.

Let $\nu_i(A)$ be the number of singular vertices with level i in $\mathcal{R}(A)$ and let m be the largest number for which $\nu_i(A) \neq 0$. Then $\nu(A) = (\nu_1(A), \ldots, \nu_m(A))$ is referred to as the *level characteristic* of A.

Let $\kappa = (K_1, \ldots, K_k)$ correspond to the Frobenius normal form of an eventually nonnegative matrix A. Let $\rho = \rho(A)$ and $m =\text{index}_\rho(A)$. For $j \in \langle m+1 \rangle$, set

$$M_j = \cup\{\ K_l \mid \text{the level of } K_l \text{ is } m+1-j \text{ in } \mathcal{R}(\rho I - A)\ \}.$$

Then $\mu = (M_1, M_2, \ldots, M_m, M_{m+1})$ is referred to as the *level partition of A with respect to the eigenvalue $\rho(A)$*, M_q is referred to as a *level set*, and A_μ is referred to as a *level form of A*. We note that our subscripting matches that of [6], but is different from [11]. Our definition also differs from that of [11] in that it includes the nonsingular classes.

In addition, we see that for any $j \in \langle m \rangle$, M_j can be further partitioned into two (not necessarily nonempty) subsets. We set

$$L_{2j} = \cup\{\ K_l \mid \text{the level of } K_l \text{ is } m+1-j \text{ in } \mathcal{R}(\rho I - A) \text{ and } \rho(A_{K_l K_l}) = \rho\}$$

and

$$L_{2j-1} = M_j \setminus L_{2j}.$$

Notice then that:

(i) $\rho(A_{L_{2j-1}L_{2j-1}}) < \rho$.

(ii) $A_{L_{2j}L_{2j}}$ is the direct sum of blocks whose spectral radius is ρ.

(iii) $A_{L_{2j-1}L_{2j}} = 0$.

We set $L_{2m+1} = M_{m+1}$ and refer to $\Lambda = (L_1, L_2, \ldots, L_{2m+1})$ as the *split–level partition of A with respect to the eigenvalue $\rho(A)$*. We will refer to L_q as a *split–level set.* Notice that A_μ and A_Λ are block lower triangular. We say that A_Λ is in *split–level form.* We refer to each $A_{L_{2j}L_{2j}}$ as a *lower level* and to each $A_{L_{2j-1}L_{2j-1}}$ as an *upper level.*

1.3 Historical Results

Much of the information presented in this thesis finds its roots with the Perron-Frobenius theorem(s), stated below ([7] 8.4.4 and 8.4.6, [1] Theorem 2.20).

Theorem 1.1 [Perron–Frobenius Theorem] Suppose $A \geq 0$ is irreducible. Then

(a) $\rho(A)$ is a simple eigenvalue of A,

(b) there exists a positive integer h such that $\pi(A) = \rho(A)\mathbb{Z}_h$,

(c) $\sigma(A) = e^{\frac{2\pi i}{h}}\sigma(A)$,

(d) if $h > 1$, then there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{h-1h} \\ A_{h1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where the zero blocks along the diagonal are square, and

(e) there exists a positive vector x such that $Ax = \rho(A)x$.

Remark 1.2 Recall that $\pi(A)$ is a multiset, so part (b) above implies that each peripheral eigenvalue is simple.

Remark 1.3 If $A \geq 0$ is reducible, then the peripheral spectrum of A must be a union of complete sets of roots of unity multiplied by $\rho(A)$.

In 1975, Uriel Rothblum [12] showed that there exists a basis of semipositive vectors for the generalized eigenspace corresponding to the spectral radius of reducible nonnegative matrices. We should note that Richman and Schneider proved similar results independently in [11]. Rothblum's result given in Theorem 1.4 below takes advantage of the combinatorial structure of a nonnegative matrix. Much can be learned by investigating the reduced graph of a matrix, especially when we take into account the Perron eigenvalue associated with each vertex in the reduced graph. We will use subscripts of vectors to represent coordinates and superscrits for enumeration. Rothblum's theorem is as follows:

Theorem 1.4 ([12] Theorem 3.1) Let $A \geq 0$ be square with $\rho = \rho(A)$, $\nu = \text{index}_\rho(A)$, and m basic classes. Let $\hat{A} = A - \rho I$. Then

(a) The algebraic eigenspace of A has a collection $x^1, \ldots, x^m$ of semipositive vectors such that $x_i^j \gg 0$ if and only if i has access to the j^{th} basic class of A, $j = 1, \ldots, m$, and any such collection is a basis for the algebraic eigenspace of A.

(b) The index of A is the length of its longest path.

(c) There is a generalized eigenvector x of A having the largest set of positive coordinates among all generalized eigenvectors, and for $n = 0 \ldots, \nu - 1$, $\hat{A}^n x > 0$. Furthermore, $(\hat{A}^n x)_i \gg 0$ if and only if i has access to some basic class in at least $n+1$ steps.

We offer an example to illustrate the previous theorem in addition to some of the definitions given in Section 1.2.

Example 1.5 Let

$$A = \left[\begin{array}{c|c|c|c|cc} 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 5 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 2 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 4 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 0 & 3 & 2 \end{array}\right].$$

Notice that A is in Frobenius Normal Form and has 5 classes, 3 basic classes, and $\rho(A) = 4$. Consider the reduced graph of A :

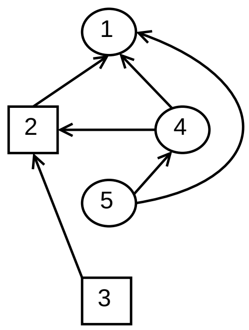

Figure 1: Reduced Graph of A

where circles denote basic classes and squares nonbasic classes. From the graph, we easily determine the level of each class, summarized below:

The Classes of A and Their Levels		
Class	Level	$\rho(A_{K_i})$
$K_1 = \{1\}$	3	4 (Basic)
$K_2 = \{2\}$	2	1 (Nonbasic)
$K_3 = \{3\}$	0	2 (Nonbasic)
$K_4 = \{4\}$	2	4 (Basic)
$K_5 = \{5, 6\}$	1	4 (Basic)

Then from Theorem 1.4, we have that the dimension of the generalized eigenspace corresponding to $\rho(A) = 4$ is 3 since A has 3 basic classes (note that this is also the size of the largest Jordan block in the Jordan form of A with eigenvalue $\rho(A) = 4$). Moreover, if we can find generalized eigenvectors of the form

$$x^1 = \begin{bmatrix} * \\ * \\ * \\ * \\ * \\ * \end{bmatrix}, x^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \\ * \\ * \end{bmatrix}, x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ * \end{bmatrix}$$

where $*$ denotes a positive entry, then the set $\{x^1, x^2, x^3\}$ will be a basis for the algebraic eigenspace of A. We may obtain an eigenvector of A with zero-nonzero pattern given by x^3 by attaching the unique (up to scalar multiplication) positive eigenvector of A_{K_5} to the zero vector of length 4. We find that

$$x^1 = \begin{bmatrix} 1 \\ 5/3 \\ 5/3 \\ 1 \\ 52/75 \\ 1 \end{bmatrix}, x^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 3 \\ 3 \end{bmatrix}, x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

are of the prescribed form and satisfy $x^3 \in null(A - 4I)$, $x^2 \in null[(A - 4I)^2]$, $x^1 \in null[(A - 4I)^3]$ and so form a basis for the algebraic eigenspace of A. Then we also have that x^1 is the generalized eigenvector having the most positive entries of all generalized eigenvectors of A.

Also, notice that, if we let $\kappa = (K_1, K_2 \cup K_4, K_5, K_3)$, then

$$A_\kappa = \left[\begin{array}{c|cc|cc|c} 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & 3 & 2 & 0 \\ \hline 0 & 2 & 0 & 0 & 0 & 2 \end{array}\right]$$

is in level form.

We see from the previous theorem and example that the reduced graph of a matrix can tell us a great deal about the spectral properties of that matrix. Richman and Schneider investigated the relationship between the reduced graph, height characteristic and level characteristic of M-matrices in [11]. Corollary 4.5 in [11] states that the height characteristic majorizes the level characteristic:

Corollary 1.6 [11, Corollary 4.5] Let B be an M-matrix with height characteristic $(\omega_1, \omega_2, \ldots, \omega_m)$ and level characteristic $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. Then

(a) $\omega_1 + \ldots + \omega_m = s = \lambda_1 + \ldots + \lambda_m$,

(b) $\omega_1 + \ldots + \omega_p \leq s - m + p$ for $p = 1 \ldots m$,

(c) $\omega_1 + \ldots + \omega_p \geq \lambda_1 + \ldots + \lambda_p$ for $p = 1 \ldots m$.

Recall that an M-matrix is a matrix of the form $B = rI - A$ where $A \geq 0$ and $r \geq \rho(A)$. Then $index_0(B) = index_{\rho(A)}(A)$ where $B = \rho(A)I - A$. So if A is a nonnegative matrix, the height characteristic of $A - \rho(A)I$ majorizes the level characteristic of $A - \rho(A)I$. We will extend this idea to the other peripheral eigenvalues in Chapter 2.

Example 1.7 Let

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}.$$

Notice that $\rho(A) = 3$. Consider the reduced graph of A below (again with circles denoting basic classes and squares nonbasic). Note that this is the same as the reduced graph of $A - 3I$ (where circles denote singular classes and squares nonsingular).

From Theorem 1.4 we know that the dimension of the algebraic eigenspace is 4 (since there are 4 basic classes) and that the length of the level and height characteristics will be 3 (length of the longest path). We see from the graph that the level characteristic of

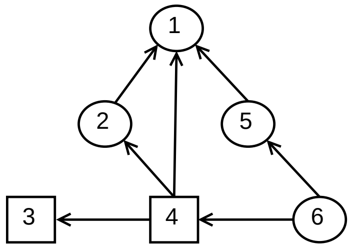

Figure 2: Reduced Graph of A

$A - 3I$ is $(1, 2, 1)$ since there is one basic vertex with level one, two with level two, and one with level three.

It can be verified that $rank(A - 3I) = 4$, $rank((A - 3I)^2) = 3$, and $rank((A - 3I)^3) = 2$. Hence, the height characteristic for $A - 3I$ is $(2, 1, 1)$ and we see that the level characteristic is indeed majorized by the height characteristic.

Notice that A has a positive generalized eigenvector since every class of A has access to some basic class of A.

Chapter 2

The Peripheral Jordan Form of a Nonnegative Matrix

In this chapter, we investigate the Jordan form corresponding to the peripheral spectrum of a nonnegative matrix. As mentioned in Section 1, it is known that if ν and η are two sequences of nonnegative integers, then there exists a nonnegative matrix A with height characteristic η, corresponding to the spectral radius, and level characteristic ν, corresponding to the spectral radius (with entries rewritten in decreasing order), if and only if ν is majorized by η ([6] Theorem 3.3 and [11] Corollary 4.5). In this chapter, we extend this property to all eigenvalues in the peripheral spectrum and offer necessary and sufficient conditions, based on the level and hight characteristics, for a multiset $\mathcal{J}$ of Jordan blocks to correspond to the peripheral spectrum of a nonnegative matrix.

2.1 Necessary and Sufficient Conditions on $\mathcal{J}$

We begin with a necessary and sufficient condition offered by McDonald in [9], called the extended Tam-Schneider condition.

Definition 2.1 Let $\mathcal{J}$ be a self–conjugate collection of Jordan blocks all of whose eigenvalues have modulus 1. Let m be the size of the largest Jordan block in $\mathcal{J}$. We say $\mathcal{J}$ satisfies the *extended Tam–Schneider condition* provided that there is a sequence of multisets $\mathcal{J}_1, \ldots, \mathcal{J}_m$ such that

(i) $J_m(1) \in \mathcal{J}$ and $\mathcal{J}_m = \mathcal{J}$.

(ii) $\mathcal{J}_1$ is a collection of 1×1 Jordan blocks which can be partitioned into complete sets of roots of unity.

(iii) For any $2 \leq j \leq m$, if we enumerate the blocks in $\mathcal{J}_j$ as $J^{(1)}, J^{(2)}, \ldots J^{(r)}$ and create the sets

$$S_j = \{(t, \lambda) | \lambda \text{ is an eigenvalue of } J^{(t)} \in \mathcal{J}_j, \text{ where } J^{(t)} \text{ is } j \times j\},$$

and

$$T_j = \{(t, \lambda) | \lambda \text{ is an eigenvalue of } J^{(t)} \in \mathcal{J}_j, \text{ where } J^{(t)} \text{ is } p \times p \text{ with } p < j\},$$

then there is a set $U_j \subseteq T_j$ so that

(a) $\mathcal{J}_{j-1}$ can be formed from $\mathcal{J}_j$ by removing the first row and column from each Jordan block in $\mathcal{J}_j$ labelled with an element appearing as a first coordinate in $S_j \cup U_j$ and leaving all other Jordan blocks the same. Note that if we remove the first row and column of a 1×1 block, then we simply remove the block itself.

(b) The second coordinates of $S_j \cup U_j$ can be partitioned into complete sets of roots of unity.

McDonald showed (in [9], Theorem 3.5) that a self–conjugate multiset $\mathcal{J}$ of Jordan blocks (all of whose eigenvalues have modulus 1) corresponds to the peripheral spectrum of a nonnegative matrix if and only if $\mathcal{J}$ satisfies the extended Tam-Schneider condition.

We seek to offer conditions equivalent to the extended Tam–Schneider condition which incorporate the idea of level sets and which lead nicely to an algorithm that determines when a multiset of Jordan blocks corresponds to the peripheral spectrum of a nonnegative matrix. What follows in the rest of this section offers just that, with the main result given in Theorem 2.9.

It is well known that the peripheral spectrum of a nonnegative matrix A is a union of complete sets of roots of unity (multiplied by $\rho(A)$). Hence, if a multiset $\mathcal{J}$, all of whose eigenvalues have modulus 1, corresponds to the peripheral spectrum of a nonnegative matrix, then $\sigma(\mathcal{J})$ must partition into complete sets of roots of unity. This partition can be determined based on the following lemma and its proof.

Lemma 2.2 If a multiset S can be partitioned into complete sets of roots of unity, then the partition is unique.

Proof: Note that the number of times 1 is listed in S determines the number of sets in the partition of S into complete sets of roots of unity. We induct on this number. If S partitions into one complete set of roots of unity, the result holds. Suppose S partitions into t sets of roots of unity. Suppose each element of S is written in the form $e^{\frac{2\pi ik}{n}}$ where $\frac{k}{n} < 1$ and $\gcd(k,n) = 1$. Let $\alpha = \max\{\frac{k}{n} : e^{\frac{2\pi ik}{n}} \in S\}$. Note that $\alpha = \frac{m-1}{m}$ for some $m \in \mathbb{N}$. Then $\mathbb{Z}_m$ must be a set in the partition of S into complete sets of roots of unity. Otherwise there is an integer $p > 1$ such that $\mathbb{Z}_{pm}$ is a set in the partition so $e^{\frac{2\pi i(pm-1)}{pm}} \in S$, but $\frac{pm-1}{pm} > \frac{m-1}{m}$, a contradiction. Then $S \setminus \mathbb{Z}_m$ partitions into complete sets of roots of unity and by the inductive hypothesis, this partition is unique. Hence, the partition of S into complete sets of roots of unity is unique. $\square$

Notice that the above proof suggests an algorithm for determining whether or not a set S partitions into complete sets of roots of unity, and if so, determining the sets in the partition. This is discussed further in the following section where an algorithm for determining whether or not a multiset of Jordan blocks correspondes to the peripheral spectrum of a nonnegative matrix is outlined. The relationship between the height characteristic and the level characteristic of a nonnegative matrix with respect to the Perron eigenvalue has played an important role in the study of reducible nonnegative matrices. Here, we illustrate how these ideas can be generalized to the entire peripheral spectrum.

Definition 2.3 Suppose $A \geq 0$. Let $\mu = (M_1, M_2, \ldots, M_m, M_{m+1})$ be the level partition of A with respect to $\rho = \rho(A)$. For each $\lambda \in \pi(A)$, let $\nu_i(\lambda)$ be the number of times λ occurs as an eigenvalue of $A_{M_iM_i}$. Then the *$\lambda-$level characteristic of A (with respect to ρ)* is the sequence $\nu_{\lambda,\rho}(A) = (\nu_1(\lambda), \ldots, \nu_m(\lambda))$ (note that $\nu_{m+1}(\lambda) = 0$ so it need not be included).

The corollary below follows easily from Lemma 3.2 of [9], which we restate here.

Lemma 2.4 [[9] Lemma 3.2] Let A be a nonnegative matrix. Set $m = \mathrm{index}_{\rho(A)}(A)$ and let $(L_1, L_2, \ldots, L_{2m+1})$ be the split–level partition of A with respect to $\rho(A)$. Set

$$P_j = \bigcup_{q=2(m+1-j)}^{2m+1} L_q$$

and let $\lambda \in \pi(A)$. Then for $j = 2, \ldots, m$, the Jordan form of λ for $A_{P_jP_j}$ can be produced from the Jordan form of λ for $A_{P_{j-1}P_{j-1}}$ by increasing the size of a select number of Jordan blocks by one, and adding copies of $J_1(\lambda)$.

Corollary 2.5 Let $A \geq 0$ with $\rho(A) = \rho$. Then, for each $\lambda \in \pi(A)$, $\hat{\nu}_{\lambda,\rho}(A) \preceq \eta_\lambda(A)$.

As stated in the above definition and corollary, the peripheral eigenvalues of a nonnegative matrix A must be distributed among the levels of A such that the majorization condition in Corollary 2.5 is satisfied. We are interested in the question of whether or not there exists a nonnegative matrix with peripheral spectrum corresponding to a given multiset $\mathcal{J}$ of Jordan blocks. Corollary 2.5 asserts that the eigenvalues of $\mathcal{J}$ must partition into level sets satisfying the majorization criterion. The definition below will allow us to gather information about a partition of a multiset of eigenvalues.

Definition 2.6 Suppose $L_1, \ldots, L_k$ are multisets of eigenvalues. For each $\lambda \in \bigcup_{i=1}^k L_i$, let $\nu_\lambda(L_i)$ be the number of times λ is listed in L_i. Then $\nu_\lambda(L_1, L_2, \ldots, L_k)$ is defined to be the sequence $(\nu_\lambda(L_1), \nu_\lambda(L_2), \ldots, \nu_\lambda(L_k))$.

The following lemmas will be used in the proof of the main theorem.

Lemma 2.7 Let α and β be decreasing sequences of length k satisfying $\alpha \preceq \beta$. Then $\alpha_j \geq \beta_k$ for $j = 1 \ldots k$.

Proof: $\Sigma_{i=1}^k \alpha_i = \Sigma_{i=1}^k \beta_i$ and $\Sigma_{i=1}^{k-1} \alpha_i \leq \Sigma_{i=1}^{k-1} \beta_i$ implies $\alpha_j \geq \alpha_k \geq \beta_k$. $\square$

Lemma 2.8 Suppose α and β are decreasing sequences of length k satisfying $\alpha \preceq \beta$. Let

$$\tilde{\alpha}(j) = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k)$$

and

$$\tilde{\beta}(j) = (\tilde{\beta}_1, \ldots, \tilde{\beta}_{k-1})$$

where

$$\tilde{\beta}_i = \beta_i - \max\{0, \alpha_j - \Sigma_{t=i+1}^k \beta_t\}.$$

Then $\tilde{\alpha}(j) \preceq \tilde{\beta}(j)$.

Proof: If $s < j$, then $\alpha_s = [\tilde{\alpha}(j)]_s$ and $\beta_s = \tilde{\beta}_s$. If $s \geq j$, then $\sum_{i=1}^s [\tilde{\alpha}(j)]_i = -\alpha_j + \sum_{i=1}^{s+1} \alpha_i \leq -\alpha_j + \sum_{i=1}^{s+1} \beta_i \leq \sum_{i=1}^s \tilde{\beta}_i$. Also, $\Sigma_{i=1}^{k-1} [\tilde{\alpha}(j)]_i = -\alpha_j + \Sigma_{i=1}^k \alpha_i = -\alpha_j + \Sigma_{i=1}^k \beta_i = \Sigma_{i=1}^{k-1} \tilde{\beta}_i$. $\square$

We are now ready to state and prove our main theorem. Note that we state our theorem for the case where $\rho(A) = 1$. If $\rho(A) \neq 1$, consider the matrix $\frac{1}{\rho(A)} A$.

Theorem 2.9 Let $\mathcal{J}$ be a self conjugate multiset of Jordan blocks all of whose eigenvalues have modulus 1. Let m be the size of the largest Jordan block in $\mathcal{J}$ with eigenvalue 1. Then the following are equivalent.

(i) $\mathcal{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix.

(ii) $\sigma(\mathcal{J}) = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_m$ where $\bigcup$ represents a multiset union and

> **(a)** Each L_i partitions into complete sets of roots of unity and
>
> **(b)** For each $\lambda \in \sigma(\mathcal{J})$, $\hat{\nu}_\lambda(L_1, \ldots, L_m) \preceq \eta_\lambda(\mathcal{J})$.

Proof: $(i) \Rightarrow (ii)$: Suppose $\mathcal{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix A. Let $(M_1, \ldots, M_{m+1})$ be the level partition of A with respect to 1. Let $L_i = \pi(A_{M_i M_i})$ for $i = 1 \ldots m$. Note that $\bigcup_{i=1}^m L_i = \pi(A) = \sigma(\mathcal{J})$. By [9] Lemma 3.1 (iii), each L_i is a union of complete sets of roots of unity. By Corollary 2.5, $\hat{\nu}_{\lambda,\rho}(A) = \hat{\nu}_\lambda(L_1, \ldots, L_m) \preceq \eta_\lambda(A) = \eta_\lambda(\mathcal{J})$.

$(ii) \Rightarrow (i)$: Set $\mathcal{J}_m = \mathcal{J}$ and let sets S_m and T_m be defined as in Definition 2.1. For each $\lambda \in \sigma(\mathcal{J})$, set $\eta_\lambda = \eta_\lambda(\mathcal{J}_m)$. Notice that $(\hat{\nu}_\lambda)_j \leq (\hat{\nu}_\lambda)_1 \leq (\eta_\lambda)_1$ for all $j \in \langle m \rangle$, so the number of λ in L_m is less than or equal to the number of Jordan blocks in $\mathcal{J}$ with eigenvalue λ. Using Lemma 2.7, $(\hat{\nu}_\lambda)_j \geq (\eta_\lambda)_m$, so the number of λ in L_m is at least the number of Jordan blocks of size m in $\mathcal{J}$. Hence, $U_m \subset T_m$ may be chosen so that the second coordinates of the elements of $S_m \bigcup U_m$ are the elements of L_m. If there is a choice, we choose the largest Jordan block(s) to be represented in U_m. Create $\mathcal{J}_{m-1}$ by removing one row and one column from each Jordan block represented in $S_m \bigcup U_m$ and leaving all others unchanged.

By Lemma 2.8, for each $\lambda \in \sigma(\mathcal{J})$, $\hat{\nu}_\lambda(L_1, \ldots, L_{m-1}) \preceq \eta_\lambda(\mathcal{J}_{m-1})$ so $U_{m-1} \subset T_{m-1}$ may be chosen so that the second coordinates of the elements of $S_{m-1} \bigcup U_{m-1}$ are the elements of L_{m-1}. We continue in this manner, noting that for each $\lambda \in \sigma(\mathcal{J})$ and $j = 1 \ldots (m-1)$, $\hat{\nu}_\lambda(L_1, \ldots, L_{m-j}) \preceq \eta_\lambda(\mathcal{J}_{m-j})$ provided that we remove rows and columns from the largest remaining Jordan block(s) when there is a choice.

Then $\mathcal{J}$ satisfies the extended Tam-Schneider condition so by [9] Theorem 3.5, $\mathcal{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix. $\square$

Remark 2.10 Part (ii) (b) of Theorem 2.9 is equivalent to saying that no two eigenvalues in an L_i come from the same Jordan block in $\mathcal{J}$.

Remark 2.11 Suppose $\mathcal{J}$ satisfies the extended Tam-Schneider condition. Then there exists a nonnegative matrix

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

in level form with Jordan form corresponding to $\mathcal{J}$. Moreover, from the proof of Theorem 3.5 in [9], we may construct A so that each subdiagonal block A_{ij}, $i > j$, is positive and has arbitrarily large entries.

The previous remark will become useful when constructing (eventually) nonnegative matrices in Chapter 3.

Observation 2.12 Suppose $\mathcal{J}$ satisfies the extended Tam-Schneider condition. Then, as shown in the proof of [9] Theorem 3.5, a nonnegative matrix A in level form can be constructed such that $\pi(A_{M_qM_q})$ consists of the second coordinates of $S_i \bigcup U_i$ for $i \in \langle m \rangle$ where $\mu = (M_1, \dots, M_{m+1})$ is the level partition of A with respect to $\rho(A)$ and $q = m+1-i$. From the proof of Theorem 2.9, L_i can be chosen to consist of the second coordinates of $S_i \bigcup U_i$. Hence, $\pi(A_{M_qM_q}) = L_i$ where $q = m+1-i$ for $i \in \langle m \rangle$.

The above observation leads to the following corollary.

Corollary 2.13 Let

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

be a nonnegative matrix in level form, all of whose eigenvalues have modulus 1. Then, for any permutation τ of $\langle m \rangle$, there exists a $B \geq 0$ similar to A such that the level form of B is

$$B = \begin{bmatrix} B_{11} & 0 & \ldots & 0 \\ B_{21} & B_{22} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \ldots & B_{mm} \end{bmatrix}.$$

and $\sigma(B_{ii}) = \sigma(A_{\tau(i)\tau(i)})$ for $i = 1, \ldots, m$.

Proof: Let $\mathcal{J}$ be the collection of Jordan blocks corresponding to the Jordan form of A. Let $L_i = \sigma(A_{qq})$ where $q = m + 1 - i$. Then by [9][Lemma 3.1, Lemma 3.2] we have m sets $L_1, L_2, \ldots, L_m$ satisfying the conditions in Theorem 2.9 (ii). But the sets $L_{\tau(1)}, L_{\tau(2)}, \ldots, L_{\tau(m)}$ also satisfy the conditions in Theorem 2.9 (ii), so we may construct a nonnegative matrix B with Jordan form corresponding to $\mathcal{J}$ and $\sigma(B_{qq}) = \pi(B_{qq}) = L_{\tau(i)} = \sigma(A_{\tau(q)\tau(q)})$ where $q = m + 1 - i$, $i = 1, \ldots, m$. Since A and B both have Jordan form corresponding to $\mathcal{J}$, A and B are similar. $\square$

We will refer to the condition stated in Theorem 2.9 (ii)(b) as the majorization condition. Suppose $\mathcal{J}$ is a multiset of Jordan blocks (all of whose eigenvalues have modulus 1) and m is the size of the largest Jordan block in $\mathcal{J}$. Note that it is possible to find a partition $L_1, \ldots, L_m$ of $\sigma(\mathcal{J})$ such that the majorization condition is satisfied. As the next example illustrates, it may also be possible to partition the eigenvalues of $\mathcal{J}$ into m sets, each of which is a union of complete sets of roots of unity. However, it may not be possible to achieve both requirements with the *same* partition of $\sigma(\mathcal{J})$, and hence $\mathcal{J}$ does not correspond to the peripheral Jordan form of a nonnegative matrix.

Example 2.14 Consider the following multiset of Jordan blocks whose spectrum partitions into $\mathbb{Z}_6 \cup \mathbb{Z}_{10} \cup \mathbb{Z}_{15}$.

$$\mathcal{J} = \{J_2(1), J_2(e^{\frac{2\pi i}{2}}), J_2(e^{\frac{2\pi i}{3}}), J_2(e^{\frac{4\pi i}{3}}), J_2(e^{\frac{2\pi i}{5}}), J_2(e^{\frac{8\pi i}{5}}), J_1(1), J_1(e^{\frac{2\pi i}{6}}), J_1(e^{\frac{10\pi i}{6}}),$$

$$J_1(e^{\frac{2\pi i}{10}}), J_1(e^{\frac{6\pi i}{10}}), J_1(e^{\frac{8\pi i}{10}}), J_1(e^{\frac{12\pi i}{10}}), J_1(e^{\frac{14\pi i}{10}}), J_1(e^{\frac{18\pi i}{10}}), J_1(e^{\frac{2\pi i}{15}}), J_1(e^{\frac{4\pi i}{15}}),$$

$$J_1(e^{\frac{8\pi i}{15}}), J_1(e^{\frac{12\pi i}{15}}), J_1(e^{\frac{14\pi i}{15}}), J_1(e^{\frac{16\pi i}{15}}), J_1(e^{\frac{18\pi i}{15}}), J_1(e^{\frac{22\pi i}{15}}), J_1(e^{\frac{26\pi i}{15}}), J_1(e^{\frac{28\pi i}{15}})\}.$$

It was shown in [9] Example 3.7 that $\mathcal{J}$ does not correspond to the peripheral spectrum of a nonnegative matrix. We see that the eigenvalues partition into complete sets of roots of unity, but these sets of roots cannot be partitioned into $m = 2$ levels so that the majorization condition is satisfied. Note that the partitions of $\sigma(\mathcal{J})$ into 2 sets, each of which is a union of complete sets of roots of unity are as follows:

(a) $L_1 = \mathbb{Z}_{15}$, $L_2 = \mathbb{Z}_{10} \bigcup \mathbb{Z}_6$ or

(b) $L_1 = \mathbb{Z}_{10}$, $L_2 = \mathbb{Z}_{15} \bigcup \mathbb{Z}_6$ or

(c) $L_1 = \mathbb{Z}_6$, $L_2 = \mathbb{Z}_{15} \bigcup \mathbb{Z}_{10}$.

Then $\eta_\lambda(\mathcal{J}) = (1,1)$ does not majorize $\hat{\nu}_\lambda(L_1, L_2) = (2,0)$ for $\lambda = e^{\frac{2\pi i}{2}}$, $e^{\frac{2\pi i}{3}}$, and $e^{\frac{2\pi i}{5}}$, respectively.

Note that it is possible to partition $\sigma(\mathcal{J})$ into 2 sets, L_1 and L_2 such that, for each $\lambda \in \sigma(\mathcal{J})$, $\hat{\nu}_\lambda(L_1, L_2) \preceq \eta_\lambda$. For example, consider $L_1 = \mathbb{Z}_{15} \bigcup \{e^{\frac{2\pi i}{2}}\}$ and $L_2 = \mathbb{Z}_{10} \bigcup \mathbb{Z}_6 \setminus \{e^{\frac{2\pi i}{2}}\}$. However, each L_i is not a union of complete sets of roots of unity.

2.2 An Algorithmic Approach

In this section, we present a recursive branch and bound algorithm based on Theorem 2.9 that will determine whether or not a multiset of eigenvalues $\mathcal{J}$ corresponds to the peripheral Jordan form of a nonnegative matrix. If $\mathcal{J}$ does correspond to the peripheral Jordan form of a nonnegative matrix, two vectors, R and N are returned indicating that a partition of $\sigma(\mathcal{J})$ into sets $L_1, \ldots, L_m$ can be formed by placing $\mathbb{Z}_{R_i}$ in L_{N_i}. We will refer to N as the assignment vector (as each entry of N assigns a set of roots to one of the sets $L_1, \ldots, L_m$.) Pseudocode for the main function is as follows:

LEVEL_PERIPH($\mathcal{J}$)

$R \leftarrow$PARTITION($\mathcal{J}$)

$N \leftarrow [1]$,

$N \leftarrow$ PLACE_ROOTS($R, N, \mathcal{J}$)

return R, N

The function PARTITION determines whether or not $\sigma(\mathcal{J})$ can be partitioned into complete sets of roots of unity. If so, the vector R is returned. The function works recursively as indicated in the proof of Lemma 2.2:

PARTITION($\mathcal{J}$)

while $\sigma(\mathcal{J}) \neq \emptyset$

  $n \leftarrow \max\{n : e^{\frac{2\pi ik}{n}} \in \sigma(\mathcal{J})\}$

  if $\mathbb{Z}_n \subset \sigma(\mathcal{J})$

$\sigma(\mathcal{J}) \leftarrow \sigma(\mathcal{J}) \setminus \mathbb{Z}_n$

$R \leftarrow [R, n]$

else

Eigenvalues do not partition into complete sets of roots of unity.

$\mathcal{J}$ does not correspond to the peripheral spectrum of a nonnegative matrix.

return

return R

The function PLACE_ROOTS is a recursive function that determines whether or not the sets of roots of unity specified in R can be placed in m sets satisfying the majorization condition. If so, the vector N is returned. We begin with $N = [N_1] = [1]$ indicating that we will place $\mathbb{Z}_{R_1}$ in L_1. We then place $\mathbb{Z}_{R_2}$ in either L_1 (if the majorization condition can still be satisfied) or L_2 (otherwise). We continue in this manner. After Z_{R_j} has been placed in some L_i, we place $Z_{R_{j+1}}$ in L_k where k is the smallest integer in $\langle m\rangle$ such that $\sum_{i=1}^{j}[\hat{\nu}_\lambda(L_1, \ldots, L_m)]_i \leq \sum_{i=1}^{j}[\eta_\lambda]_i$ for $j \in \langle m\rangle$. If no such k exists, then we prune the current branch of the search tree and proceed to the next vertex. Note that $\sum_{i=1}^{m}[\hat{\nu}_\lambda(L_1, \ldots, L_m)]_i = \sum_{i=1}^{m}[\eta_\lambda]_i$ once all sets of roots of unity have been placed. A sample search tree for the case when $\sigma(\mathcal{J}) = \mathbb{Z}_{R_1} \bigcup \mathbb{Z}_{R_2} \bigcup \mathbb{Z}_{R_3} \bigcup \mathbb{Z}_{R_4}$ and $m = 3$ is given in Figure 3.

Since the numbering of the sets $L_1, \ldots, L_m$ is arbitrary, placement of $\mathbb{Z}_{R_1}$ in any other L_i need not be considered. Moreover, we only consider N satisfying $N_i \leq 1 + \max_{j<i} N_j$ for $i \in \langle m\rangle$ since we do not assign a set of roots to L_i if none have yet been assigned to L_{i-1}. Also, at least one set of roots must be assigned to each L_i, otherwise the majorization condition will be violated for $\lambda = 1$. So if $\max(N) < m - \text{length}(R) + \text{length}(N)$,

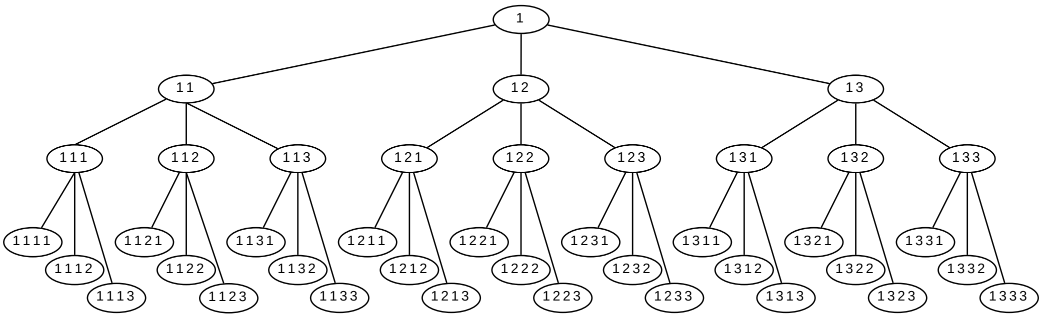

Figure 3: Search tree for $m = 3$ and $\sigma(\mathcal{J}) = \mathbb{Z}_{R_1} \bigcup \mathbb{Z}_{R_2} \bigcup \mathbb{Z}_{R_3} \bigcup \mathbb{Z}_{R_4}$

then N is not a valid assignment vector so we move to the next assignment vector. This allows us to prune many branches of the search tree quickly. For example, the only vertices remaining after pruning the tree in Figure 3 are shown in Figure 4. The pseudocode for PLACE_ROOTS is given below.

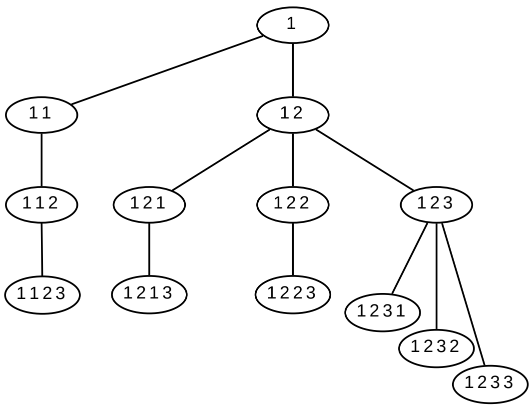

Figure 4: Pruned search tree for $m = 3$ and $\sigma(\mathcal{J}) = \mathbb{Z}_{R_1} \bigcup \mathbb{Z}_{R_2} \bigcup \mathbb{Z}_{R_3} \bigcup \mathbb{Z}_{R_4}$

PLACE_ROOTS$(R, N, \mathcal{J})$

if $N_1 = 2$

    No level assignments satisfy the majorization condition.

    $\mathcal{J}$ does not correspond to the peripheral spectrum of a nonnegative matrix.

    return

$k \leftarrow$length(N)

if $N_k > 1 + \max_{j<k} N_j$ or $\max_{j\leq k} N_j < m-$length(R)+length(N)

    $N \leftarrow$BYPASS_CHILDREN(N)

    PLACE_ROOTS($R, N, \mathcal{J}$)

if majorization condition can be/is satisfied

    if length(N) =length(R)

        return R, N

    else $N \leftarrow$ NEXT_VERTEX(N)

else $N \leftarrow$BYPASS_CHILDREN(N)

PLACE_ROOTS($R, N, \mathcal{J}$)

NEXT_VERTEX is a function which proceeds to the next vertex in the pre-ordered search tree. BYPASS_CHILDREN is used when the current N is not a valid assignment vector (so no assignment vector beginning with the entries of N will be valid). In this case, we move to the next vertex in the search tree which does not begin with the current assignment vector N. Once all of the vertices in the search tree have been considered or pruned, both NEXT_VERTEX and BYPASS_CHILDREN return the assignment vector $N = [2]$ which indicates that $\mathcal{J}$ does not correspond to the peripheral Jordan form of a nonnegative matrix. Both functions are modeled after the pseudocode given in [8], pages 107-108.

When a multiset $\mathcal{J}$ does correspond to the peripheral spectrum of a nonnegative matrix, the algorithm J2NN computes a nonnegative matrix with the prescribed Jordan form. J2NN applies a series of similarity transformations to the direct sum of blocks in $\mathcal{J}$ to arrive at a nonnegative matrix.

The MATLAB codes for level_periph and J2NN are given in Appendix A and Appendix B, respectively.

Chapter 3

The Jordan Form of an Eventually Nonnegative Matrix

We now turn our focus to the entire (as opposed to just the peripheral) Jordan form of an eventually nonnegative matrix. It is known that the peripheral Jordan form of an eventually nonnegative matrix must satisfy the same conditions as the peripheral Jordan form of a nonnegative matrix (the conditions given in Theorem 2.9). We begin our discussion by investigating the Jordan forms and Frobenius normal forms of powers of an eventually nonnegative matrix. Then, we look at the structure of an eventually nonnegative matrix in terms of split-level sets and generalize split-level sets to define the component level sets of an eventually nonnegative matrix. Along the way, we give multiple sufficient conditions, escalating in complexity, for a multiset of Jordan blocks to correspond to the Jordan form of an (eventually) nonnegative matrix. Necessary and sufficient conditions on the Jordan form of an eventually nonnegative matrix are given in Section 3.4. Lastly, with a goal of finding an algorithm that will determine whether or not a given multiset, $\mathcal{J}$, of Jordan blocks corresponds to the spectrum of an eventually nonnegative matrix in mind, we offer examples which discuss some of the issues/difficulties that arise in writing such an algorithm.

3.1 Jordan Forms and Frobenius Normal Forms of Powers of A

In this section, we compare the Jordan form of a matrix, A, with the Jordan forms of the powers of A. We also investigate the relationship between the Frobenius normal forms of A and the Frobenius normal forms of powers of A.

First, suppose $\lambda \neq 0$ and $n \in \mathbb{N}$. Consider the Jordan block $J_n(\lambda)$. Notice that $[J_n(\lambda)]^k$ is the matrix with all diagonal entries equal to λ^k and j^{th} subdiagonal entries equal to $\binom{k}{k-j}\lambda^{k-j}$ if $j \leq k$ and 0 otherwise. Hence, the Jordan form of $[J_n(\lambda)]^k$ is $J_n(\lambda^k)$, so in some sense the Jordan form is preserved as we power up the matrix; the Jordan block sizes do not change and the eigenvalues are just powers of the original eigenvalues. In other words, for a matrix A and $\lambda \neq 0$, $J_n(\lambda) \in \mathcal{J}(A)$ implies $J_n(\lambda^k) \in \mathcal{J}(A^k)$, and the blocks occur the same number of times. Note that it is possible for $\lambda, \mu \in \sigma(A)$ with $\lambda \neq \mu$, but $\lambda^k = \mu^k$. For example, let $\lambda = 1$, $\mu = -1$, and k be an even natural number. Then $\lambda^k = \mu^k$, but $\lambda \neq \mu$.

However, we do not have this property (of preservation of Jordan block sizes) for the Jordan blocks with eigenvalue zero. Notice that $[J_n(0)]^k$ is just the $n \times n$ zero matrix when $k \geq n$. So the Jordan form for the eigenvalue zero is not preserved when we power up the matrix unless all zero eigenvalue blocks were originally 1×1.

Carnochan Naqvi and McDonald discuss the structure of eventually nonnegative matrices having $\text{index}_0(A) \leq 1$ in [2]. In particular, it is shown that if A is an eventually nonnegative matrix with $\text{index}_0(A) \leq 1$, then a large prime number g can be chosen so that if $\kappa = (K_1, K_2, \ldots, K_k)$ is the ordered partition such that A^g_κ is in Frobenius normal form, then A_κ is also in Frobenius normal form (see [2], Theorem 3.4). Elhashash and

Szyld give the set N_A of natural numbers for which the above property and those listed below hold in [4].

- $A^k \geq 0$ for all $k \geq g$ and
- for any $\lambda, \mu \in \sigma(A)$, $\lambda^g = \mu^g$ if and only if $\lambda = \mu$.

In keeping with the convention used by Carnochan Naqvi and McDonald in [2], we will simply observe that, if A is an eventually nonnegative matrix, then there is a large prime g for which the above properties hold. We encourage the interested reader to see [4] for a complete characterization of all of the values of g for which the above properties hold. Notice that a consequence of this is that we may obtain the Jordan form for all nonzero eigenvalues of A from the Jordan form of A^g.

The properties discussed thus far in this section can be applied to eventually nonnegative matrices for which $\text{index}_0(A) > 1$ by considering the following decomposition theorem given by Zaslavsky and Tam.

Theorem 3.1 ([16], Theorem 3.6) Let A be an $n \times n$ complex matrix.

(a) The matrix A can be expressed uniquely in the form $B + C$, where $\text{index}_0(B) \leq 1$ and C is nilpotent such that $BC = CB = 0$.

(b) With B, C as given in (a), the collection of nonsingular Jordan blocks in $\mathcal{J}(A)$ is the same as the collection of nonsingular Jordan blocks in $\mathcal{J}(B)$, and the collection of singular Jordan blocks in $\mathcal{J}(A)$ can be obtained from $\mathcal{J}(C)$ by removing from it r $J_1(0)$'s where r =rank(B).

(c) With B as given in (a), A is eventually nonnegative if and only if B is eventually nonnegative.

Remark 3.2 Notice that part (c) in the previous theorem follows from the observation that $A^k = B^k$ for $k \geq \text{index}_0(A)$.

We end this section by offering the following example to illustrate some of the properties discussed thus far and to make further observations corresponding to the peripheral Jordan form of an eventually nonnegative matrix as compared to the peripheral Jordan form of a nonnegative matrix with respect to the structure of the matrix. We note that the matrices used in this example were presented in [16].

Example 3.3 Let $A = B + C$ where

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

First, we verify that the conditions in Theorem 3.1 are met. The reader may verify that $BC = CB = 0$. The Jordan forms corresponding to A, B, and C are given by:

- $\mathcal{J}(A) = \{J_1(2), J_1(2), J_2(0)\}$ (so $\text{index}_0(A) = 2$),
- $\mathcal{J}(B) = \{J_1(2), J_1(2), J_1(0), J_1(0)\}$ (so $\text{index}_0(B) = 1$), and
- $\mathcal{J}(C) = \{J_2(0), J_1(0), J_1(0)\}$ (so $\text{index}_0(C) = 2$).

Notice that $\mathcal{J}(A)$ can indeed be obtained from $\mathcal{J}(B)$ and $\mathcal{J}(C)$ in a manner described in Theorem 3.1 (b). Also notice that B is (eventually) nonnegative and hence A is eventually nonnegative since $A^k = B^k$ for $k \geq 2 = \text{index}_0(A)$. Also notice that A is irreducible, but $\rho(A) = 2$ is a repeated eigenvalue of A. This fact illustrates a difference in the combinatorial structure of nonnegative as opposed to eventually nonnegative matrices as the spectral radius cannot be a repeated eigenvalue of an irreducible nonnegative matrix. Notice, however, that for $k \geq 2$, A^k is reducible. It is essentially the contribution of the zero eigenvalue blocks that result in A being irreducible, and once we power up the matrix enough so that the nilpotent part is zero (again, we just need to take a power larger than $\mathrm{index}_0(A)$), we obtain a reducible matrix.

3.2 Component Level Sets

The idea of level sets was first introduced by Richman and Schneider in [11]. Level sets have since been used by many authors. McDonald defines split-level sets in [9]. We extend this idea to define the component level sets for an eventually nonnegative matrix. We will then use these component level sets to discuss combinatorial properties of eventually nonnegative matrices. We first offer an example to motivate the definition.

Example 3.4 Let

$$A = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 \\ 0 & A_{52} & 0 & A_{54} & A_{55} & 0 \\ 0 & 0 & 0 & 0 & A_{65} & A_{66} \end{bmatrix}$$

be a nonnegative matrix in Frobenius Normal form. Suppose $\rho(A_{55}) = \rho(A_{66}) = \rho(A) = \rho_1$, $\rho(A_{11}) = \rho(A_{44}) = \rho_2$, and $\rho(A_{22}) = \rho(A_{33}) = \rho_3$ where $\rho_1 > \rho_2 > \rho_3$. The reduced graph of A is then

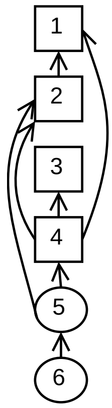

Figure 5: The reduced graph of A from which we determine the split-level partition of A

where circles denote basic classes and squares nonbasic. We see that the split-level partition of A is given by

$$A = \left[\begin{array}{cccc|c|c} A_{11} & 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 \\ \hline 0 & A_{52} & 0 & A_{54} & A_{55} & 0 \\ \hline 0 & 0 & 0 & 0 & A_{65} & A_{66} \end{array}\right] \tag{3.1}$$

Recall that $\rho(A) = \rho(A_{55}) = \rho(A_{66})$ and we refer to each of the blocks A_{55} and A_{66} as a lower level. There is only one nonempty upper level in A, namely the upper left block in the block partition given in 3.1. This upper level has spectral radius ρ_2 and is itself a nonnegative matrix. Hence, we may consider the split-level partition of this upper level with respect to ρ_2 to further partition the matrix A. Below is the reduced graph of this upper level:

We see that further partitioning A based on the split-level partition of its upper left

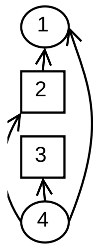

Figure 6: The reduced graph of an upper level of A.

submatrix from the partition in 3.1 results in the following partition:

$$A = \left[\begin{array}{c|cc|c|c|c} A_{11} & 0 & 0 & 0 & 0 & 0 \\ \hline A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 & 0 \\ \hline A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 \\ \hline 0 & A_{52} & 0 & A_{54} & A_{55} & 0 \\ \hline 0 & 0 & 0 & 0 & A_{65} & A_{66} \end{array}\right] \tag{3.2}$$

Notice that only one "upper" level was formed at this step, namely the $A_{22} \oplus A_{33}$ block. This upper level is an eventually nonnegative matrix with spectral radius ρ_3 and we may consider its split-level partition and further partition A accordingly. The split level partition of $A_{22} \oplus A_{33}$ is trivial, so, in this case, A is not partitioned further. We have now considered all spectral radii. The partition of A given in 3.2 corresponds to the component level partition of A. Each diagonal block will be referred to as a component level (of A).

The ideas illustrated in the above example are formalized in the definition below.

Definition 3.5 Let A be an eventually nonnegative matrix with $P(A) = \{\rho_1, \rho_2, \dots, \rho_t\}$ where $\rho_1 > \rho_2 > \cdots > \rho_t$. Suppose $\Lambda_1 = (L_1^{(1)}, L_2^{(1)}, \dots, L_{m_1}^{(1)})$ is the split-level partition of A with respect to ρ_1. For $k = 2, \dots, t$, replace each $L_j^{(k-1)}$ in Λ_{k-1} for which $\rho(A_{L_j^{(k-1)} L_j^{(k-1)}}) = \rho_k$ with the split-level partition of $A_{L_j^{(k-1)} L_j^{(k-1)}}$ with respect to ρ_k to obtain $\Lambda_k = (L_1^{(k)}, L_2^{(k)}, \dots, L_{m_k}^{(k)})$. Delete all empty sets from $\Lambda_t = (L_1^{(t)}, L_2^{(t)}, \dots, L_{m_t}^{(t)})$ to obtain $\mathcal{C} = (C_1, C_2, \dots, C_\ell)$ Then $\mathcal{C}$ is the *component level partition of* A. We refer to each C_j as a *component level set* and refer to each $A_{C_j C_j}$ as a *component level of* A. We say that $A_{\mathcal{C}}$ is in *component level form.*

Remark 3.6 Note that in the definition above, we use the indices from the original matrix A when determining the split-level partition of submatrices.

Recall that the necessary and sufficient conditions (see Theorem 2.9) for a multiset $\mathcal{J}$ of Jordan blocks to correspond to the peripheral Jordan form of a nonnegative matrix rely on the notion of level sets. Similarly, we will see that component levels play an important role in the necessary and sufficient conditions on the Jordan form of an eventually nonnegative matrix. Since the component levels play such an important role, we discuss some related properties below.

Remark 3.7 If $\mathcal{C} = (C_1, C_2, \dots, C_\ell)$ is the component-level partition of an eventually nonnegative matrix A, then each $A_{C_j C_j}$ is a direct sum of irreducible eventually nonnegative matrices with a common spectral radius.

Then the corollary below follows from Theorem 5.1 of [16].

Corollary 3.8 Let A be an eventually nonnegative matrix and g a large prime. Suppose A^g has component level partition $\mathcal{C} = (C_1, C_2, \dots, C_\ell)$. For each $j \in \langle \ell \rangle$, let $\mathcal{J}_j$ be the multiset of Jordan blocks corresponding to the Jordan form of $A^g_{C_jC_j}$. Then each $\mathcal{J}_j$ is a union of self-conjugate Frobenius multisets, each with spectral radius $\rho(A^g_{C_jC_j})$.

Remark 3.9 Since, for a large prime g, the Jordan form of an eventually nonnegative matrix A can be obtained from the Jordan form of A^g, we know that the Jordan form of A must also satisfy the combinatorial properties indicated in the above corollary.

In much the same way as we construct the spectrum of levels to create a nonnegative matrix in Chapter 2, we will attempt to construct multisets which will correspond to the Jordan form of component levels in order to construct eventually nonnegative matrices in the following sections. This task is more complicated for a variety of reasons. We offer the following example to point out that we really do need to consider the component levels of A^g as opposed to just looking at the structure of A itself to gain insight into the Jordan form of the matrix A.

Example 3.10 Consider the matrices given in Example 3.3:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

and $A = B + C$. Also consider $\hat{A} = \hat{B} + \hat{C}$ where

$$\hat{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } \hat{C} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We note that $\hat{A}$, $\hat{B}$, and $\hat{C}$ appeared in [2]. Now, A and $\hat{A}$ are both irreducible eventually nonnegative matrices (and so each have only one level). However, $\rho(A) = \rho(\hat{A}) = 2$ is a repeated root of both A and $\hat{A}$. The Jordan forms of A and $\hat{A}$ are given by

- $\mathcal{J}(A) = \{J_1(2), J_1(2), J_2(0)\}$
- $\mathcal{J}(\hat{A}) = \{J_2(2), J_2(0)\}$.

So we see that, even though A and $\hat{A}$ both have the same level form, they have different Jordan forms corresponding to the eigenvalue 2. Thus, in this case, we cannot predict the Jordan forms based on the level structures of A and $\hat{A}$. Instead, consider

$$A^2 = B^2 = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \text{ and } \hat{A}^2 = \hat{B}^2 = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \end{bmatrix}.$$

Now we see a significant difference between the structure of A^2 and that of $\hat{A}^2$. The matrix A^2 still consists of a single (component) level, but is reducible. There are two (component) levels in $\hat{A}^2$, and $\hat{A}^2$ is also reducible. We pause momentarily to comment that, in this example, component levels coincide exactly with the levels, so we will just write 'level' instead of 'component level' for the remainder of the discussion. We now see that the Jordan form corresponding to the spectral radius is in agreement with the level form for both matrices. For the matrix A^2, there is only one level, and the size of the largest Jordan block in $\mathcal{J}(A^2)$ with eigenvalue $\rho(A^2) = \rho(A)^2 = 4$ is 1×1. Analagously, $\hat{A}^2$ has two levels, and the size of the largest Jordan block with eigenvalue $\rho(\hat{A}^2) = 4$ is 2×2. In both cases, the spectral radius is a simple eigenvalue of each irreducible block (which is not the case for the original matrices A and $\hat{A}$). For the matrices A and $\hat{A}$ and any $g \geq 2$, A^g and $\hat{A}^g$ will satisfy the properties discussed above for A^2 and $\hat{A}^2$. We see that the component level form of A^g is a better indicator of the Jordan form of A than A itself (similarly for $\hat{A}^g$).

3.3 Sufficient Conditions on $\mathcal{J}$

Thus far, we have discussed combinatorial properties of eventually nonnegative matrices beginning with the eventually nonnegative matrix. We would now like to begin with a multiset $\mathcal{J}$ of Jordan blocks and determine whether or not $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix. We will use the idea of component levels and attempt to construct the Jordan form of each component level of the desired eventually nonnegative matrix from the given $\mathcal{J}$. This approach is similar to that taken with the peripheral Jordan form of a nonnegative matrix discussed in Chapter 2, and in fact the condition on the peripheral Jordan form of an eventually nonnegative matrix is exactly the same as that on the peripheral Jordan form of a nonnegative matrix (see Theorem 2.9). We will present several sufficient conditions on $\mathcal{J}$ along with an algorithm that has been implemented in MATLAB that determines if $\mathcal{J}$ satisfies one of the sufficient conditions. We comment that the sufficent conditions given in this section are actually for $\mathcal{J}$ to correspond to the Jordan form of a nonnegative matrix. We include these conditions here as opposed to in Chapter 2 since the conditions build up to a necessarry and sufficient condition discussed in Section 3.4.

Since we are now interested in the Jordan form of an eventually nonnegative matrix (as opposed to the peripheral Jordan form of a nonnegative matrix), we must consider multisets $\mathcal{J}$ of Jordan blocks whose eigenvalues may not have a common magnitude. We begin with a definition which is a slight generalization of the extended Tam-Schneider condition.

Definition 3.11 Let $\mathcal{J}$ be a multiset of Jordan blocks all of whose eigenvalues have modulus ρ. Let m be a positive integer. Then $\mathcal{J}$ *can be leveled into* m *multisets* $L_1, \ldots, L_m$ provided $\sigma(\mathcal{J}) = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_m$ where $\bigcup$ represents a multiset union and

(a) Each L_i is nonempty and partitions into complete sets of roots of unity multiplied by ρ and

(b) For each $\lambda \in \sigma(\mathcal{J})$, $\hat{\nu}_\lambda(L_1, \ldots, L_m) \preceq \eta_\lambda(\mathcal{J})$.

Notice the similarity of the above definition to the conditions in Theorem 2.9; the only difference is that m is not set to be the size of the largest Jordan block in $\mathcal{J}$ with eigenvalue ρ. If $\mathcal{J}$ is a multiset of Jordan blocks which corresponds to the Jordan form of an eventually nonnegative matrix, it is known that $\mathcal{J}^{\rho(\mathcal{J})}$ can be leveled into m multisets where m is the size of the largest Jordan block in $\mathcal{J}$ with eigenvalue $\rho(\mathcal{J})$. Notice that the following sufficient condition for $\mathcal{J}$ to correspond to the spectrum of a nonnegative matrix is a corollary to Theorem 2.9.

Corollary 3.12 Let $\mathcal{J}$ be a multiset of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \rho_2, \ldots, \rho_t\}$ where $\rho_1 > \rho_2 > \cdots > \rho_t$. For $j \in \langle t \rangle$, let m_j be the size of the largest Jordan block in $\mathcal{J}$ with eigenvalue ρ_j. Then there exists a nonnegative matrix A with Jordan form corresponding to $\mathcal{J}$ if each $\mathcal{J}^{\rho_j}$ can be leveled into m_j multisets.

Proof: By Theorem 2.9, for $j \in \langle t \rangle$, $\mathcal{J}^{\rho_j}$ corresponds to the peripheral spectrum of a nonnegative matrix, call it A_j. Then $A = A_1 \oplus A_2 \oplus \cdots \oplus A_t$ is a nonnegative matrix with Jordan form corresponding to $\mathcal{J}$. $\square$

The condition given in the above corollary can be generalized slightly to obtain another sufficient condition on the Jordan form of a nonnegative matrix. We take advantage of the idea of component levels and show that we can adjust our choice of m_j's from the previous corollary to allow more possibilities. The following definition is given in order to specify the number of multisets into which we may level each $\mathcal{J}^{\rho_j}$.

Definition 3.13 Let $\mathcal{J}$ be a multiset of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \ldots, \rho_t\}$ where $\rho_1 > \rho_2 > \ldots > \rho_t \geq 0$. Suppose, for each $j \in \langle t \rangle$,

$$\mathcal{J}(\rho_j) = \{J_{n_1(j)}(\rho_j), J_{n_2(j)}(\rho_j), \ldots, J_{n_{\ell_j}(j)}(\rho_j)\}$$

where $n_1(j) \geq n_2(j) \geq \cdots \geq n_{\ell_j}(j)$. Then set $m_1 = n_1(1)$ and for each $2 \leq j \leq t$, set

$$m_j = \sum_{k=1}^{(m_1+m_2+\cdots+m_{j-1}+1)} n_k(j).$$

We denote the vector $(m_1, m_2, \ldots, m_t)$ by $m(\mathcal{J})$.

The following lemma will be used in the proof of Theorem 3.15 and is useful in the construction of (eventually) nonnegative matrices.

Lemma 3.14 Let N be a nonnegative matrix in Frobenius normal form such that each irreducible block N_{jj} of N satisfies $\rho(N_{jj}) = \rho(N)$. Let A be a nonnegative matrix such that $\rho(A) > \rho(N)$ and suppose A has positive left and right eigenvectors corresponding to $\rho(A)$. Then we have the following.

(i) There exists a nonsingular matrix Q satisfying

$$Q(N \oplus A)Q^{-1} = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} N & 0 \\ P_1 & A \end{bmatrix}$$

where X is a positive matrix and P_1 is a positive matrix whose entries can be made arbitrarily large with an appropriate choice of X.

(ii) There exists a nonsingular matrix R satisfying

$$R(A \oplus N)R^{-1} = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ P_2 & N \end{bmatrix}$$

where Y is a positive matrix and P_2 is a positive matrix whose entries can be made arbitrarily large with an appropriate choice of Y.

Proof: (i): We induct on the number m of irreducible classes of N. Suppose $m = 1$. Then N is an irreducible nonnegative matrix and has a positive left eigenvector corresponding to $\rho(N)$, call it y. Let x be the positive right eigenvector of A. Let $Q_1 = \begin{bmatrix} I & 0 \\ -\alpha x y^T & I \end{bmatrix}$ where α is a large positive scalar. Then

$$Q_1(N \oplus A)Q_1^{-1} = \begin{bmatrix} I & 0 \\ -\alpha x y^T & I \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} I & 0 \\ \alpha x y^T & I \end{bmatrix} = \begin{bmatrix} N & 0 \\ \alpha A x y^T - \alpha x y^T N & A \end{bmatrix}$$

Notice that $\alpha A x y^T - \alpha x y^T N = \alpha x y^T(\rho(A) - \rho(N)) \gg 0$ and can be chosen to have arbitrarily large entries by an appropriate choice of α.

Now suppose N has $m \geq 2$ irreducible classes. Consider the block partition of N given by

$$N = \begin{bmatrix} N_1 & 0 \\ N_2 & N_3 \end{bmatrix}$$

where N_1 is a single irreducible block of N. By the inductive hypothesis, there is a $Q_2 = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ such that $Q_2(N_3 \oplus A)Q_2^{-1} = \begin{bmatrix} N_3 & 0 \\ P & A \end{bmatrix}$ where P is a positive matrix. Then notice that

$$(I \oplus Q_2)(N \oplus A)(I \oplus Q_2)^{-1} = \begin{bmatrix} N_1 & 0 & 0 \\ N_2 & N_3 & 0 \\ -XN_2 & P & A \end{bmatrix}.$$

Now, let $\hat{x}$ and $\hat{y}$ be the positive right eigenvector of A and positive left eigenvector of N_1, respectively. Set $Q_3 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\alpha_1 \hat{x}\hat{y}^T & 0 & I \end{bmatrix}$. Then

$$Q_3(I \oplus Q_2)(N \oplus A)(I \oplus Q_2)^{-1}Q_3^{-1} = \begin{bmatrix} N_1 & 0 & 0 \\ N_2 & N_3 & 0 \\ \alpha_1 A\hat{x}\hat{y}^T - \alpha_1 \hat{x}\hat{y}^T N_1 - XN_2 & P & A \end{bmatrix}$$

Notice that $\alpha_1 A\hat{x}\hat{y}^T - \alpha_1 \hat{x}\hat{y}^T N_1 - XN_2 = \alpha_1(\rho(A) - \rho(N_1))\hat{x}\hat{y}^T - XN_2$ and can be made a positive matrix with arbitrarily large entries with an appropriate choice of α_1.

The proof of (ii) is analogous. □

We are now ready to state a more general sufficient condition.

Theorem 3.15 Let $\mathcal{J}$ be a self-conjugate multiset of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \ldots, \rho_t\}$ where $\rho_1 > \rho_2 > \ldots > \rho_t \geq 0$ and $m(\mathcal{J}) = (m_1, \ldots, m_t)$. Then $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix if $\mathcal{J}^{\rho_j}$ can be leveled into m_j multisets for each $j \in \langle t \rangle$.

Proof: We induct on t. For $t = 1$, $\mathcal{J}$ can be leveled into m_1 multisets $L_1, L_2, \ldots, L_{m_1}$ and $\mathcal{J}$ corresponds to the Jordan form of a nonnegative matrix by Theorem 2.9. Moreover, the resulting nonnegative matrix has m_1 levels and the (peripheral) spectrums of the levels are given by the multisets $L_1, L_2, \ldots, L_{m_1}$.

Let $\mathcal{J}$ be a multiset of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \rho_2, \ldots, \rho_{t+1}\}$ where $\rho_1 > \rho_2 > \cdots > \rho_t$. We suppose there exists a nonnegative matrix

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m_t,1} & A_{m_t,2} & \cdots & A_{m_t,m_t} \end{bmatrix}$$

in component level form with Jordan form corresponding to $\mathcal{J} \setminus \mathcal{J}^{\rho_{t+1}}$ where the subdiagonal blocks of A are positive matrices whose entries are arbitrarily large (see Remark 2.11). Now, since $\mathcal{J}^{\rho_{t+1}}$ can be leveled into m_{t+1} multisets, there exist multisets of Jordan blocks $U_1, \ldots, U_{m_t+1}$ such that

- $\mathcal{J}_{n_j(t+1)}(\rho_{t+1}) \in U_j$ for each $j \in \langle m_t + 1 \rangle$, where the blocks in $\mathcal{J}(\rho_{t+1})$ are enumerated as in Definition 3.13, and
- each U_j satisfies the extended Tam-Schneider condition.

The collection $\{U_1, \ldots, U_{m_t+1}\}$ is obtained from $\mathcal{J}^{\rho_{t+1}}$ by block partitioning each $J \in \mathcal{J}^{\rho_{t+1}}$ and placing the resulting diagonal blocks in different U_j's. Since each U_j satisfies the extended Tam-Schneider condition, there exists a nonnegative matrix N_{jj} with Jordan form corresponding to U_j for each $j \in \langle m_t + 1 \rangle$. Moreover, we may construct a matrix

$$N = \begin{bmatrix} N_{11} & 0 & \cdots & 0 \\ N_{21} & N_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_{m_t+1,1} & N_{m_t+1,2} & \cdots & N_{m_t+1,m_t+1} \end{bmatrix}$$

where each N_{ij} is real and each N_{jj} is nonnegative as described above (we obtain real subdiagonal blocks by taking advantage of the self-conjugacy of $\mathcal{J}^{\rho_{t+1}}$). Then consider the matrix

$$B = \begin{bmatrix} N_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_{11} & 0 & 0 & \cdots & 0 \\ N_{21} & 0 & N_{22} & 0 & \cdots & 0 \\ 0 & A_{21} & 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N_{m_t+1,1} & 0 & N_{m_t+1,2} & 0 & \cdots & N_{m_t+1,m_t+1} \end{bmatrix}$$

Each block in the above block partition of B is a nonnegative matrix with the exception of the N_{ij} blocks when $i > j$ and B has Jordan form corresponding to $\mathcal{J}$. We now argue that we can apply a string of similarity transformations to B to construct a nonnegative matrix. By Lemma 3.14 (i), there exists a $Q_1 = \begin{bmatrix} I & 0 \\ -X_1 & I \end{bmatrix}$ satisfying $Q_1 \begin{bmatrix} N_{11} & 0 \\ 0 & A_{11} \end{bmatrix} Q_1^{-1} = \begin{bmatrix} N_{11} & 0 \\ P_1 & A_{11} \end{bmatrix}$ where P_1 is a positive matrix with arbitrarily large entries. Then

$$(Q_1 \oplus I)B(Q_1 \oplus I)^{-1} = \begin{bmatrix} N_{11} & 0 & 0 & 0 & \cdots & 0 \\ P_1 & A_{11} & 0 & 0 & \cdots & 0 \\ N_{21} & 0 & N_{22} & 0 & \cdots & 0 \\ A_{21}X_1 & A_{21} & 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{m_t+1,1} & 0 & N_{mt+1,2} & 0 & \cdots & N_{m_t+1,M_t+1} \end{bmatrix}$$

Note that each zero block appearing in the first column of B has been replaced by a positive matrix of the form $A_{ij}X_1$. Now, by Lemma 3.14 (ii), there exists a $Q_2 = \begin{bmatrix} I & 0 \\ Y_1 & I \end{bmatrix}$ such that $Q_2 \begin{bmatrix} A_{11} & 0 \\ 0 & N_{22} \end{bmatrix} Q_2^{-1} = \begin{bmatrix} A_{11} & 0 \\ P_2 & N_{22} \end{bmatrix}$ where Y and P_2 are positive matrices. Then

$$\begin{aligned} &(I \oplus Q_2 \oplus I)(Q_1 \oplus I)B(Q_1 \oplus I)^{-1}(I \oplus Q_2 \oplus I)^{-1} \\ &= \begin{bmatrix} N_{11} & 0 & 0 & 0 & 0 & \cdots & 0 \\ P_1 & A_{11} & 0 & 0 & 0 & \cdots & 0 \\ Y_1P_1 + N_{21} & P_2 & N_{22} & 0 & 0 & \cdots & 0 \\ A_{21}X_1 & A_{21} & 0 & A_{22} & 0 & \cdots & 0 \\ N_{31} & -N_{31}Y_1 & N_{32} & 0 & N_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ N_{m_t+1,1} & -N_{mt+1,2}Y_1 & N_{m_t+1,2} & 0 & \cdots & N_{m_t+1,m_t+1} & \end{bmatrix} \end{aligned}$$

Now, $Y_1P_1 + N_{21} \gg 0$ since Y_1 and P_1 are positive and the entries of P_1 can be made arbitrarily large. At this point, all subdiagonal blocks of the first three rows are positive matrices whose entries can be made arbitrarily large. We continue in this manner, applying Lemma 3.14 and the fact that the matrices A_{ij}, $i > j$ are positive matrices with large entries to fill in the remaining subdiagonal blocks one row at a time. In this fashion, we obtain a nonnegative matrix whose Jordan form corresponds to $\mathcal{J}$. $\square$

We would like to comment that a MATLAB function is currently being written which will test a given multiset $\mathcal{J}$ of Jordan blocks against the condition given in Theorem 3.15.

3.4 Necessary and Sufficient Conditions on $\mathcal{J}$

In this section, we offer necessary and sufficient conditions on the Jordan form of an eventually nonnegative matrix which are stated as Theorem 3.23. In addition, numerous examples are given to illustrate the difficulties that arise in finding an algorithm that will determine whether or not a given multiset $\mathcal{J}$ of Jordan blocks corresponds to the Jordan form of an eventually nonnegative matrix.

In keeping with the approach taken in the previous section, we attempt to construct the Jordan form of the component levels of the desired eventually nonnegative matrix. The steps taken to do this are summarized briefly below.

1. Start with a self-conjugate multiset $\mathcal{J}$ of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \rho_2, \ldots, \rho_t\}$ where $\rho_1 > \rho_2 > \cdots > \rho_t$.

2. Check to see that $\mathcal{J}^{\rho_1}$ satisfies the extended Tam Schneider condition (conditions of Theorem 2.9). If so, we obtain multisets which correspond to the peripheral Jordan form of the levels of the desired eventually nonnegative matrix.

3. Consider $\mathcal{J}^{\rho_2}$. We must be able to assign Jordan blocks to the multisets obtained

in the previous step (the "existing" multisets) and/or to "new" multisets (which will correspond to the peripheral Jordan form of upper levels of the desired matrix) so that the updated existing multisets partition into self-conjugate Frobenius multisets with spectral radius ρ_1 and each new multiset satisfies the extended Tam Schneider condition. We must also have that the Jordan form assigned to each multiset is compatible with our original Jordan form.

4. We continue in this manner, beginning with the multisets resulting from the previous step, we add to these multisets or create new multisets between existing multisets so that new multisets satisfy the extended Tam Schneider condition and existing multisets remain unions of self-cojugate Frobenius collections.

We illustrate this process with the example below.

Example 3.16 Let

$$\mathcal{J} = \left\{ \begin{bmatrix} 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} 3e^{\frac{2\pi i}{3}} \end{bmatrix}, \begin{bmatrix} 3e^{\frac{4\pi i}{3}} \end{bmatrix}, \right.$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2i \end{bmatrix}, \begin{bmatrix} -2i \end{bmatrix}, \begin{bmatrix} 2e^{\frac{2\pi i}{5}} \end{bmatrix}, \begin{bmatrix} 2e^{\frac{4\pi i}{5}} \end{bmatrix}, \begin{bmatrix} 2e^{\frac{6\pi i}{5}} \end{bmatrix}, \begin{bmatrix} 2e^{\frac{8\pi i}{5}} \end{bmatrix},$$

$$\left. \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} e^{\frac{2\pi i}{6}} \end{bmatrix}, \begin{bmatrix} e^{\frac{10\pi i}{6}} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 1 & -i \end{bmatrix} \right\}$$

Then $P(\mathcal{J}) = \{3, 2, 1\}$ and we first check to see that $\mathcal{J}^3$ satisfies the extended Tam Schneider condition. Indeed it does, as can be seen by taking

$L_2 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}]\}$ and

$$L_1 = \{[3], [-3]\}$$

Notice that L_2 partitions into two self-conjugate Frobenius collections, one with index of cyclicity 2 and the other with index of cyclicity 3. This must remain the case as we update L_2 in subsequent steps by adding blocks with eigenvalues of smaller magnitudes. Likewise, L_1 must remain 2-cyclic when blocks are added. At this point, if $\mathcal{J}$ does correspond to the Jordan form of an eventually nonnegative matrix A, we know that A must have two levels (perhaps in addition to a zero level). L_1 and L_2 give us the peripheral Jordan form of the levels of A. Note that, in this example, L_1 and L_2 are unique up to a reordering of the multisets.

Now, we must consider $\mathcal{J}^2$. We may create three new upper levels at this point (interlaced with the existing levels) and/or update the existing L_1 and L_2. The "new" upper levels created at this point must satisfy the extended Tam-Schneider condition (and will perhaps be broken into several new component levels as we shall see). Notice that $[2i]$ and $[-2i]$ must be placed in either L_1 or L_2 since $[-2] \notin \mathcal{J}^2$ so it would be impossible for a multiset containing $[2i]$ or $[-2i]$ to satisfy the extended Tam-Schneider condition. Also, we may place $[2i]$ and $[-2i]$ in either L_1 or L_2 and we choose one of the multisets arbitrarily at this point. Notice that $2\mathbb{Z}_5 \subset \sigma(\mathcal{J}^2)$, but 5 is not an allowable index of cyclicity for either L_1 or L_2 (since neither 2 nor 3 divides 5) and hence, these fifth roots must be represented in a new upper level. Note that one arrangement that is allowable for this step is given by

$$L_2 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}]\},$$

$$U_1 = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}] \right\},$$

$L_1 = \{[3], [-3], [2i], [-2i]\}$.

Then U_1 satisfies the extended Tam-Schneider condition as required; take

$U_1^{(2)} = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\}$ and

$U_1^{(1)} = \{[2]\}$.

This results in a current assignment of Jordan blocks to component levels given by:

$C_4 = L_2 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}]\}$,

$C_3 = U_1^{(2)} = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\}$,

$C_2 = U_1^{(1)} = \{[2]\}$, and

$C_1 = L_1 = \{[3], [-3], [2i], [-2i]\}$.

Notice that the multisets have simply been relabeled to avoid the use of further subscripting as we move to the next step. We now consider $\mathcal{J}^1$. We may create up to five new multisets, each satisfying the extended Tam Schneider condition and/or update the existing component levels. There are numerous options for the placement of Jordan blocks in the existing component levels and/or new levels. Before giving a few of the options, we would like to make some observations. Since 1 only occurs once as an eigenvalue in $\sigma(\mathcal{J}^1)$, we can only create one new component level at this step. Notice that $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are each subsets of $\sigma(\mathcal{J}^1)$, and we may create a new component level with spectrum equal to either $\mathbb{Z}_2$ or $\mathbb{Z}_4$. Also notice that $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are both 2-cyclic, so could be incorporated into any of the three component levels C_1, C_2, or C_4, but not C_3 as it is 5-cyclic. Also note that C_2 has index of cyclicity 1, so we may assign all Jordan blocks from $\mathcal{J}^1$ to C_2 and the resulting multiset would still be a self-conjugate Frobenius multiset with index of cyclicity 1. However, we give a different allowable assignment for this last step below:

$$C_4 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}], [e^{\frac{2\pi i}{6}}], [-1], [e^{\frac{10\pi i}{6}}]\},$$

$$C_3 = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\},$$

$$C_2 = \{[2]\}, \text{ and}$$

$$C_1 = \left\{ [3], [-3], [2i], [-2i], [1], [-1], \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, \begin{bmatrix} -i & 0 \\ 1 & -i \end{bmatrix}, \right\}.$$

In this case, we were able to construct self-conjugate Frobenius multisets from the given $\mathcal{J}$ and so $\mathcal{J}$ does correspond to the Jordan form of an eventually nonnegative matrix. We note that the eventually nonnegative matrix that could be formed from the multisets specified above would have two levels (since there are two C_j's with spectral radius $\rho(\mathcal{J}) = 3$) and one nonempty upper level. If we consider this one nonempty upper level in its own right, it has two levels. So we have a total of four component levels. We note that it would also be possible to construct an eventually nonnegative matrix with six component levels had we used

$$C_6 = \{[1], [-1], [i], [-i]\},$$

$$C_5 = \{[2]\},$$

$$C_4 = \{[3], [-3], [3], [3e^{\frac{2\pi i}{3}}], [3e^{\frac{4\pi i}{3}}], [e^{\frac{2\pi i}{6}}], [-1], [e^{\frac{10\pi i}{6}}]\},$$

$$C_3 = \{[2]\},$$

$$C_2 = \{[3], [-3], [2i], [-2i], [i], [-i]\}, \text{ and}$$

$$C_1 = \{[2], [2e^{\frac{2\pi i}{5}}], [2e^{\frac{4\pi i}{5}}], [2e^{\frac{6\pi i}{5}}], [2e^{\frac{8\pi i}{5}}]\},$$

for example. We could also construct five component levels, but there is no way to use fewer than four component levels. We would like to note that there are many more options for the choice of component levels than the two given above in this example. We see that the choices made at each step are not unique.

Another way to view the construction of the Jordan form of component levels in the previous example is that we simply applied a permutation similarity to $\oplus\mathcal{J}$ and block partitioned the resulting matrix so that each diagonal block corresponded to a self-conjugate Frobenius multiset (in addition to other conditions pertaining to the number of diagonal blocks and placement with respect to each other). We will see in the next example that only allowing permutation similarities is not enough and we may need to allow a more general class of similarity transformations. The more general transformations that must be considered are given in Definition 3.18.

Example 3.17 Suppose $\alpha = e^{\theta i}$ for some $\theta \in (0, \frac{\pi}{2})$. Let

$$\mathcal{J} = \{J_2(2), J_2(-2), J_3(\alpha), J_1(\alpha), J_2(-\alpha), J_2(-\alpha), J_3(\overline{\alpha}), J_1(\overline{\alpha}), J_2(-\overline{\alpha}), J_2(-\overline{\alpha})\}.$$

Note that $P(\mathcal{J}) = \{1, 2\}$ and $\mathcal{J}^2$ satisfies the extended Tam-Schneider condition (take $L_1 = \{[2], [-2]\} = L_2$). Also notice $1 \notin \mathcal{J}^1$. Hence, if $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix A, then A has two irreducible classes and the Jordan form of each class must correspond to a self-conjugate Frobenius collection with index of cyclicity 2.

Let $J = \oplus\mathcal{J}$. Notice that J is 20×20, but there is no partition $\kappa = (K_1, K_2)$ of 20 for which each $J_{K_jK_j}$ corresponds to a self-conjugate Frobenius collection. However, let

$$C = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}$$

where

$$C_{11} = C_{22} = J_1(2) \oplus J_1(-2) \oplus J_2(\alpha) \oplus J_2(-\alpha) \oplus J_2(\overline{\alpha}) \oplus J_2(-\overline{\alpha})$$

and $C_{21} = [c_{ij}]$ where $c_{11} = c_{22} = c_{33} = c_{77} = 1$ and all other $c_{ij} = 0$. Then C is similar to J and each C_{jj} is in Jordan form and has Jordan form corresponding to a self-conjugate Frobenius collection. Hence, $\mathcal{J}$ does correspond to the Jordan form of an eventually nonnegative matrix.

The following definition gives the class of matrices we must consider when constructing the Jordan form of component levels given a Jordan matrix.

Definition 3.18 Let J be an $n \times n$ matrix in Jordan form and $\kappa = (K_1, \ldots, K_k)$ a partition of $\langle n \rangle$. We say that the matrix C *is a reorganization of* J *with respect to* κ provided

(i) The matrix C is similar to J.

(ii) Each C_{K_j} is in Jordan form.

(iii) C_κ is lower triangular.

(iv) $c_{ii} \neq c_{jj}$ implies $c_{ij} = c_{ji} = 0$ for all $i, j \in \langle n \rangle$.

Since we are interested in considering multisets of Jordan blocks as opposed to the direct sum of Jordan blocks, we offer the following definition of a multiset reorganization.

Definition 3.19 Let $\mathcal{J}$ be a multiset of Jordan blocks and let m be a natural number. We say that $\mathcal{J}$ *can be multiset reorganized into m multisets denoted by* $(S_1, S_2, \ldots, S_m)$ provided each S_j is a multiset of Jordan blocks and there exists a reorganization

$$C = \begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{bmatrix} = [c_{ij}]$$

of $\oplus\mathcal{J}$ such that for each $j \in \langle m \rangle$, $C_{jj} = \oplus S_j$.

The next two definitions are given as they are needed for the proof of Theorem 3.23.

Definition 3.20 ([10] Definition 4.4) Let $\mathscr{L}$ be a set of ordered partitions of $\langle n \rangle$. We will say that $\mathscr{L}$ is a *set of nested odd partitions of* n provided that

(i) $\mathscr{L}$ contains exactly one ordered partition of $\langle n \rangle$.

(ii) $\mathscr{L}$ contains exactly one ordered partition of each nonempty set which occurs in an odd position of an ordered partition in $\mathscr{L}$.

(iii) The partitions described in (i) and (ii) are the only partitions in $\mathscr{L}$.

Definition 3.21 ([10] Definition 4.6) Let J be an $n \times n$ matrix, $\kappa = (K_1, \ldots, K_k)$ an ordered partition of $\langle n \rangle$, C a reorganization of J with respect to κ, and $\mathscr{L}$ a set of nested odd partitions of n. We say that C *with the partition* κ *allows the nested odd partition* $\mathscr{L}$ provided that $\rho(C_{K_j}) \in \sigma(C_{K_j})$ for all $j \in \langle k \rangle$, and for each ordered partition $\Lambda \in \mathscr{L}$, we have the following.

Write $\Lambda = (L_1, L_2, \ldots, L_t)$ and for each $i \in \langle t \rangle$, set

$$P_i = \{j | K_j \cap L_i \neq \emptyset\}.$$

Set

$$P = \bigcup_{i=1}^{t} P_i, \quad \tau = \bigcup_{i=1}^{t} L_i, \quad \rho_\tau = \rho(C_\tau), \quad m_\tau = \mathrm{index}_{\rho_\tau}(C_\tau).$$

Then we need that

(a) $t = 2m_\tau + 1$.

(b) $L_i = \cup_{j \in P_i} K_j$.

(c) If $j \in P$ and $\rho(C_{K_j}) = \rho_\tau$, then $j \in P_i$ where i is even.

(d) If $j \in P$ and $\rho(C_{K_j}) < \rho_\tau$, then $j \in P_i$ where i is odd.

(e) If $j, \ell \in P$ with $j \neq \ell$ and $C_{K_j K_\ell} \neq 0$, then $j \in P_q$ and $\ell \in P_r$ where $q \geq r$ and if $q = r$ then q must be odd.

The following appeared as part of Theorem 3.3 of [3] and offers necessary and sufficient conditions for a multiset $\mathcal{J}$ of Jordan blocks to correspond to the Jordan form of an eventually nonnegative matrix. The characterization is based on reorganizations and the notions defined above. We note that the conditions given in part (ii) below first appeared in [10], when a certain class of matrices called seminonnegative matrices were considered. In [3], it was proven that eventually nonnegative matrices are similar to seminonnegative matrices and the theorem below followed.

Theorem 3.22 ([10] Theorem 4.10, [3] Theorem 3.3) Let J be an $n \times n$ matrix in Jordan form. Then the following are equivalent.

(i) The matrix J is similar to an eventually nonnegative matrix.

(ii) There exists an ordered partition $\kappa = (K_1, \ldots, K_k)$ of $\langle n \rangle$ and a reorganization C of J with respect to κ such that

(a) For each nonzero $\lambda \in \sigma(J)$, $C_\kappa(\bar{\lambda}) = \overline{C_\kappa(\lambda)}$.

(b) For each $j \in \langle k \rangle$, C_{K_j} corresponds to a self-conjugate Frobenius collection.

(c) C with the partition κ allows a set of nested odd partitions of $\langle n \rangle$.

We are now ready to state necessary and sufficient conditions for a multiset $\mathcal{J}$ of Jordan blocks to correspond to the Jordan form of an eventually nonnegative matrix.

Theorem 3.23 Let $\mathcal{J}$ be a multiset of Jordan blocks with $P(\mathcal{J}) = \{\rho_1, \rho_2, \ldots, \rho_t\}$ where $\rho_1 > \rho_2 > \ldots > \rho_t$. Then $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix if and only if

(a) $\mathcal{J}^{\rho_1}$ can be leveled into $m_1 = \text{index}_{\rho_1}(\mathcal{J}^{\rho_1})$ multisets denoted by $(L_1^{(1)}, L_2^{(1)}, \ldots, L_{m_1}^{(1)})$ and

(b) for $j = 2 \ldots t$, $\mathcal{J}^{\rho_j}$ can be multiset reorganized into $2m_{j-1} + 1$ multisets, some possibly empty, denoted by

$$(S_1^{(j)}, S_2^{(j)}, \ldots, S_{2m_{j-1}+1}^{(j)})$$

where, for each $k \in \langle 2m_{j-1} + 1 \rangle$,

(i) if k is even, then $L_{k/2}^{(j-1)} \bigcup S_k^{(j)}$ partitions into self-conjugate Frobenius multisets, each with spectral radius greater than ρ_j, and

(ii) if k is odd, then $S_k^{(j)}$ can be leveled into $m_{k,j}$ =index$_{\rho_j}(S_k^{(j)})$ multisets denoted by $(T_{k,1}^{(j)}, T_{k,2}^{(j)}, \ldots, T_{k,m_{k,j}}^{(j)})$.

Set $L^{(j)*} = (T_{1,1}^{(j)}, \ldots, T_{1,m_{1,j}}^{(j)}, L_1^{(j-1)} \cup S_2^{(j)}, T_{3,1}^{(j)}, \ldots, T_{3,m_{3,j}}^{(j)}, L_2^{(j-1)} \cup S_4^{(j)}, \ldots, L_{2m_{j-1}}^{(j-1)} \cup S_{m_{j-1}}^{(j)}, T_{2m_{j-1}+1,1}, \ldots, T_{2m_{j-1}+1, m_{2m_{j-1}+1,j}})$. Delete all empty sets listed in $L^{(j)*}$ and relabel the remaining multisets to obtain $L^{(j)} = (L_1^{(j)}, L_2^{(j)}, \ldots, L_{m_j}^{(j)})$. Note that m_j is the number of multisets listed in $L^{(j)}$.

Proof: ($\Leftarrow$) Suppose $\mathcal{J}$ satisfies the conditions given in (a) and (b) above. First consider the case where $\rho_t > 0$. We will show that $J = \oplus\mathcal{J}$ satisfies the conditions given in Theorem 3.22 (ii). Suppose J is $n \times n$. Note that $\mathcal{J}$ can be multiset reorganized into m_t multisets denoted by $L^{(t)} = (L_1^{(t)}, L_2^{(t)}, \ldots, L_{m_t}^{(t)})$. Hence, there exists a matrix

$$C = \begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ C_{21} & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{m_t 1} & C_{m_t 2} & \cdots & C_{m_t m_t} \end{bmatrix}$$

which is similar to J and for each $j \in \langle m_t \rangle$, we have that the Jordan form of C_{jj} is $\oplus L_j^{(t)}$. Let $\kappa = (K_1, \ldots, K_{m_t})$ be the partition of n corresponding to the block partition of C given above. Since each $L_j^{(t)}$ partitions into self-conjugate Frobenius multisets, there is a refinement $\hat{\kappa} = (\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_k)$ of the ordered partition κ such that each $C_{\hat{K}_j}$ has Jordan form corresponding to a single self-conjugate Frobenius multiset.

Construct $\mathscr{L}$, a set of ordered partitions, as follows. Suppose $K_{n_1}, K_{n_2}, \ldots, K_{n_\ell}$ are all of the elements of κ satisfying $\rho(C_{K_j}) = \rho_1$ where $n_1 < n_2 < \cdots < n_\ell$. Set

$$\Lambda_0 = \left(\bigcup_{j=1}^{n_1 - 1} K_j, K_{n_1}, \bigcup_{j=n_1+1}^{n_2-1} K_j, K_{n_2}, \ldots, \bigcup_{j=n_{\ell-1}+1}^{n_\ell - 1} K_j, K_{n_\ell}, \bigcup_{j=n_\ell+1}^{m_t} K_j \right).$$

Note that some sets listed in odd positions of Λ_0 may be empty, but are still included in the list. Place Λ_0 in $\mathscr{L}$.

Now, set $n_0 = 0$, $n_{\ell+1} = m_t+1$, and $\kappa(n_j) = (K_{n_{j-1}+1}, \ldots, K_{n_j-1})$ for each $j \in \langle \ell+1 \rangle$. Let $\rho(j) = \rho(C_{\kappa(n_j)})$. We construct ordered partitions Λ_j from $\kappa(n_j)$ in the same way that Λ_0 was constructed from κ. If $\kappa(n_j)$ is empty, do nothing. Otherwise, suppose $K_{n_1(j)}, K_{n_2(j)}, \ldots, K_{n_{\ell_j}(j)}$ are the elements of $\kappa(n_j)$ satisfying $\rho(C_{K_{n_i(j)}}) = \rho(j)$ where $n_1(j) < n_2(j) < \cdots < n_{\ell_j}(j)$. Set

$$\Lambda_j = \left(\bigcup_{i=n_{j-1}+1}^{n_1(j)-1} K_i, K_{n_1(j)}, \ldots, \bigcup_{i=n_{\ell_{j-1}}+1}^{n_{\ell_j}-1} K_i, K_{n_{\ell_j}(j)}, \bigcup_{i=n_{\ell_j}(j)+1}^{n_j-1} K_i \right).$$

Again, some sets listed in Λ_j may be empty, but are still included in the list. Place each Λ_j in $\mathscr{L}$. We proceed in this manner, creating ordered partitions of each set listed in an odd position of each Λ_j based on the associated spectral radius, and add these ordered partitions to $\mathscr{L}$. Continue until sets listed in odd positions are all empty. Note that each K_j will be a set listed in exactly one element of $\mathscr{L}$ after $\mathscr{L}$ is completely constructed. Then C allows the nested odd partition $\mathscr{L}$ and hence J satisfies the conditions of Theorem 3.22. Therefore, there exists an eventually nonnegative matrix with Jordan form corresponding to $\mathcal{J}$.

Now we consider the case where $\rho_t = 0$. From the above proof, there is an eventually nonnegative matrix A with Jordan form corresponding to $\mathcal{J} \setminus \mathcal{J}^0$. Then the matrix obtained by taking the direct sum of A with the blocks from $\mathcal{J}^0$ is an eventually nonnegative matrix with Jordan form corresponding to $\mathcal{J}$.

($\Rightarrow$) Suppose $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix A. We will show that the conditions given in Theorem 3.22 imply the conditions in parts (a) and (b) of this Theorem. Let $J = \oplus \mathcal{J}$, and suppose J is $n \times n$. Let κ be an ordered partition of $\langle n \rangle$, C a reorganization of $\mathcal{J}$ with respect to κ, and $\mathscr{L}$ a set of nested odd partitions of $\langle n \rangle$ satisfying the conditions in Theorem 3.22 (ii). We first establish some notation that will be used in this proof. For $\Lambda \in \mathscr{L}$, write $\Lambda = (M_1, M_2, \ldots, M_\ell)$ and let $\tau = \cup_{i=1}^{\ell} M_i$. We will denote $\rho(C_\tau)$ by $\rho(\Lambda)$. We will denote the multiset of Jordan blocks with eigenvalues of greatest modulus corresponding to the Jordan form of $C_{M_j M_j}$ by $\mathcal{J}^\pi(C_{M_j M_j})$.

Let $\Lambda^{(0)}$ be the unique partition in $\mathscr{L}$ satisfying $\rho(\Lambda^{(0)}) = \rho_1$ and write $\Lambda^{(0)} = (M_1, M_2, \ldots, M_\ell)$. Set $L_j^{(1)} = \mathcal{J}^\pi(C_{M_{2j}})$ for each j. Then $(L_1^{(1)}, L_2^{(1)}, \ldots, L_{m_1}^{(1)})$ satisfies the condition given in part (a) of this theorem. Define the sets $U_1^{(1)}, \ldots, U_{m_1}^{(1)}$ by $U_j^{(1)} = \mathcal{J}(C_{M_{2j}}) \backslash \mathcal{J}^\pi(C_{M_{2j}})$. Note that the indices from each M_{2j+1} all appear in other partitions in $\mathscr{L}$ and will be considered at a later step. Now, for $j = 2 \ldots, t$, choose all $\Lambda \in \mathscr{L}$ such that $\rho(\Lambda) = \rho_j$. Enumerate these partitions as $\Lambda^{(1)}, \Lambda^{(2)}, \ldots, \Lambda^{(\ell_1)}$. For k odd, set $S_k^{(j)} = \mathcal{J}^\pi(C_{\Lambda^{(\frac{k+1}{2})}})$ and write $\Lambda^{(\frac{k+1}{2})} = (M_1^{(k)}, M_2^{(k)}, \ldots, M_{s_k}^{(k)})$. Set $T_{k,r}^{(j)} = \mathcal{J}^\pi(C_{M_{2r}^{(k)}})$ and $U_{k,r}^{(j)} = \mathcal{J}(C_{M_{2r}^{(k)}}) \setminus \mathcal{J}^\pi(C_{M_{2r}^{(k)}})$. Again, note that the $M_j^{(k)}$'s, where j is odd are considered at a later step as they are further partitioned and the partition appears in $\mathscr{L}$. Now, let $L^{(j)*} = (T_{1,1}^{(j)}, \ldots, T_{1,m_{1,j}}^{(j)}, L_1^{(j-1)} \cup \mathcal{J}^\pi(C_{U_1^{(j-1)}}), T_{3,1}^{(j)}, \ldots, T_{3,m_{3,j}}^{(j)}, L_2^{(j-1)} \cup \mathcal{J}^\pi(C_{U_2^{(j-1)}}), \ldots, L_{2m_{j-1}}^{(j-1)} \cup \mathcal{J}^\pi(C_{U_m^{(j-1)}}), T_{2m_{j-1}+1,1}, \ldots, T_{2m_{j-1}+1,m_{2m_{j-1}+1,j}})$. Delete the empty sets from $L^{(j)*}$ and relabel the entries to obtain $L^{(j)} = (L_1^{(j)}, L_2^{(j)}, \ldots, L_{m_j}^{(j)})$. Relabel the $U_i^{(j)}$'s conformally.

$\square$

We conclude this section by investigating examples which point out some of the issues that must be considered to obtain an algorithm which tests a given $\mathcal{J}$ against these necessary and sufficient conditions.

Given a multiset of eigenvalues, it can quickly be determined whether or not the multiset partitions into complete sets of roots of unity. If the multiset does partition into complete sets of roots of unity, the partition is unique (see Lemma 2.2). However, it is more difficult to determine whether or not the multiset partitions into shifted complete sets of roots of unity and, if it does, we are not guarenteed uniqueness. This poses a problem in finding an algorithm which determines whether or not a given multiset, $\mathcal{J}$, of Jordan blocks corresponds to the spectrum of an eventually nonnegative matrix as it is necessary that each $\sigma(\mathcal{J}^{\rho_j})$ (where $P(\mathcal{J}) = \{\rho_1, \rho_2, \ldots, \rho_t\}$) partitions into shifted complete sets of roots of unity multiplied by ρ_j. The next few examples illustrate some issues that arise in determining whether or not $\sigma(\mathcal{J})$, where $\mathcal{J}$ is a given multiset of Jordan blocks, partitions into shifted complete sets of roots and how these partitions relate to the desired eventually nonnegative matrix.

Example 3.24 In this example, we illustrate a difficulty pertaining to the representation of eigenvalues in $\sigma(\mathcal{J})$, where $\mathcal{J}$ is a multiset of Jordan blocks. We will see that a single eigenvalue must be viewed as two different shifted roots. Let

$$\begin{aligned} \mathcal{J} &= \{J_2(2), J_1(2e^{\frac{2\pi i}{3}}), J_1(2e^{\frac{4\pi i}{3}}), J_1(2e^{\pi i}), J_2(e^{\frac{3\pi i}{4}}), J_2(e^{\frac{5\pi i}{4}}), \\ & \qquad J_1(e^{\frac{\pi i}{4}}), J_1(e^{\frac{7\pi i}{4}}), J_1(e^{\frac{\pi i}{12}}), J_1(e^{\frac{23\pi i}{12}}), J_1(e^{\frac{17\pi i}{12}})\}. \end{aligned}$$

Notice that $P(\mathcal{J}) = \{2, 1\}$. In order for $\mathcal{J}$ to correspond to the Jordan form of an eventually nonnegative matrix, $\mathcal{J}^2$ must satisfy the extended Tam-Schneider condition. Notice that the size of the largest Jordan block in $\mathcal{J}^2$ is 2 and that $\sigma(\mathcal{J}^2) = 2\mathbb{Z}_2 \bigcup 2\mathbb{Z}_3$. Setting $L_1 = 2\mathbb{Z}_2$ and $L_2 = 2\mathbb{Z}_3$, we see that $\mathcal{J}^2$ does indeed satisfy the extended Tam-Schneider condition. Notice that $1 \notin \sigma(\mathcal{J}^1)$, and hence $\sigma(\mathcal{J}^1)$ does not contain a complete set of roots of unity. Therefore, in order for $\mathcal{J}$ to correspond to the Jordan form of an eventually nonnegative matrix, it is necessary that $\sigma(\mathcal{J}^1)$ partitions into two sets, S_1 and S_2, where S_1 is 2-cyclic and S_2 is 3-cyclic (since L_1 is 2-cyclic and L_2 is 3-cyclic). The only such partition is given by $S_1 = e^{\frac{\pi i}{4}}\mathbb{Z}_4$ and $S_2 = e^{\frac{\pi i}{12}}\mathbb{Z}_3 \bigcup e^{\frac{-\pi i}{12}}\mathbb{Z}_3$. Then we can construct an eventually nonnegative matrix

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

in Frobenius normal form where $\sigma(A_{11}) = L_1 \bigcup S_1$ and $\sigma(A_{22}) = L_2 \bigcup S_2$.

We would like to point out a difficulty in determining the sets S_1 and S_2. Note that $e^{\frac{3\pi i}{4}} \in e^{\frac{\pi i}{4}}\mathbb{Z}_4$ and $e^{\frac{3\pi i}{4}} \in e^{\frac{\pi i}{12}}\mathbb{Z}_3$. So, on one hand we have considered the eigenvalue $e^{\frac{3\pi i}{4}}$ as a 4^{th} root of unity shifted by $\theta_1 = \frac{\pi}{4}$ (note $e^{\frac{3\pi i}{4}} = e^{(\theta_1 + \frac{\pi}{2})i}$) and on the other hand as a 3^{rd} root of unity shifted by $\theta_2 = \frac{\pi}{12}$ (note $e^{\frac{3\pi i}{4}} = e^{(\theta_2 + \frac{2\pi}{3})i}$). In general, for any natural number n and any eigenvalue λ, λ can be viewed as a shifted n^{th} root.

Example 3.25 In this example, we illustrate a difficulty in partitioning the eigenvalues from a multiset $\mathcal{J}$ of Jordan blocks into cyclic multisets. Let $\theta \in (0, \frac{\pi}{4})$ and

$$\mathcal{J} = \{[3], [-3], [2], [2i], [-2], [-2i], \begin{bmatrix} e^{i\theta} & 0 \\ 1 & e^{i\theta} \end{bmatrix}, \begin{bmatrix} e^{-i\theta} & 0 \\ 1 & e^{-i\theta} \end{bmatrix},$$

$$[-e^{i\theta}], [-e^{i\theta}], [-e^{-i\theta}]\}, [-e^{-i\theta}]\}.$$

We will see that $\mathcal{J}$ does correspond to the Jordan form of an eventually nonnegative matrix

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

in split-level form. However, even though $2\mathbb{Z}_4 \subseteq \sigma(\mathcal{J})$, we show that it is impossible for $2\mathbb{Z}_4$ to be a subset of either $\sigma(A_{11})$ or $\sigma(A_{22})$.

Now, we are interested in determining whether or not $\mathcal{J}$ corresponds to the Jordan form of an eventually nonnegative matrix. Notice that $P(\mathcal{J}) = \{1, 2, 3\}$ and that $\mathcal{J}_3$ trivially satisfies the extended Tam-Schneider condition. We proceeed by attempting to build multisets which will correspond to the spectra of the component levels (in this example simply the split-levels) of an eventually nonnegative matrix. Begin by setting

$$\begin{aligned} L_2^{(1)} &= \emptyset, \\ L_1^{(1)} &= \{3, -3\}, \\ L_0^{(1)} &= \emptyset \end{aligned} .$$

Note that $L_1^{(1)}$ is 2-cyclic and that $\sigma(\mathcal{J}_2) = 2\mathbb{Z}_4$. Now, we must assign each of the elements of $\sigma(\mathcal{J}_2)$ to one of the multisets $L_2^{(1)}, L_1^{(1)}$, or $L_0^{(1)}$ so that each of the resulting multisets ($L_2^{(2)}, L_1^{(2)}$, and $L_0^{(2)}$) corresponds to a self-conjugate Frobenius collection. One way to do this is to set

$$\begin{aligned} L_2^{(2)} &= \emptyset, \\ L_1^{(2)} &= \{3, -3\}, \\ L_0^{(2)} &= \{2, 2i, -2, -2i\} \end{aligned} .$$

Now, $\sigma(\mathcal{J}_1) = e^{i\theta}\mathbb{Z}_2 \bigcup e^{i\theta}\mathbb{Z}_2 \bigcup e^{-i\theta}\mathbb{Z}_2 \bigcup e^{-i\theta}\mathbb{Z}_2$. Since all elements of $\mathcal{J}_1$ are purely complex, we must assign each element to either $L_1^{(1)}$ or $L_0^{(1)}$ in order for the resulting multisets to correspond to Frobenius sets. Notice that $\mathcal{J}_1$ is not 2-cyclic (nor 4-cyclic), and hence, we cannot assign all elements of $\mathcal{J}_1$ to a single multiset. Also notice that there is no 4-cyclic subset of $\sigma(\mathcal{J}_1)$, and so we cannot assign any elements to $L_0^{(2)}$. Therefore, we must adjust our original choices of $L_2^{(2)}, L_1^{(2)}$, and $L_0^{(2)}$. Instead, consider

$$\begin{aligned} \hat{L}_2^{(2)} &= \emptyset, \\ \hat{L}_1^{(2)} &= \{3, -3, 2i, -2i\}, \\ \hat{L}_0^{(2)} &= \{2, -2\} \end{aligned} .$$

Now we have two 2-cyclic sets to work with instead of the original 2-cyclic and 4-cyclic sets and can set

$$\begin{aligned} L_2^{(3)} &= \emptyset, \\ L_1^{(3)} &= \{3, -3, 2i, -2i, e^{i\theta}, -e^{i\theta}, e^{-i\theta}, -e^{-i\theta}\}, \\ L_0^{(3)} &= \{2, -2, e^{i\theta}, -e^{i\theta}, e^{-i\theta}, -e^{-i\theta}\} \end{aligned} \cdot$$

We see that each $L_j^{(3)}$ corresponds to a self-conjugate Frobenius collection. Therefore, there does exist an eventually nonnegative matrix A with Jordan form corresponding to $\mathcal{J}$. The sets $L_2^{(3)}, L_1^{(3)}$, and $L_0^{(3)}$ give the spectra corresponding to the split-level partition of A with respect to $\rho(A) = 3$.

Chapter 4

Concluding Remarks

A great deal is known about the combinatorial structure of nonnegative and eventually nonnegative matrices, some of which has been addressed in this paper. We have written neccessary and sufficient conditions for a multiset of Jordan blocks to correspond to the peripheral Jordan form of an (eventually) nonnegative matrix. In addition, an algorithm has been written and implemented in MATLAB which tests a multiset against these conditions and returns a nonnegative matrix with the prescribed Jordan form if possible.

We have also offered necessary and sufficient conditions for a multiset of Jordan blocks to correspond to the Jordan form of an eventually nonnegative matrix. These conditions are more complicated than those on the peripheral Jordan form, and we have offered examples which point out some of the difficulties in writing an algorithm to test these conditions. In the future, we would like to investigate these difficulties further and write an algorithm. We would like to know exactly what the allowable multiset reorganizations are for a given multiset of Jordan blocks. That is, given a Jordan matrix J, what are the possible Jordan forms of the diagonal blocks of a block lower triangular matrix which is similar to J?

The question as to how we represent eigenvalues should also be addressed. As mentioned in Chapter 3, if λ is a complex number, we can view λ as a shifted n^{th} root for any $n \in \mathbb{N}$, and we may need to view λ as both a shifted n_1^{th} root and as a shifted n_2^{th} for some $n_1 \neq n_2$ even though they may share a Jordan block. We also know that we may have to split up a complete set of roots of unity, viewing, say, $\mathbb{Z}_4$ as two sets of (shifted) 2^{nd} roots as seen in Example 3.25.

Even knowing the allowable multiset reorganizations and an effective way to represent eigenvalues, the question as to how exactly to write an algorithm testing the conditions in Theorem 3.23 may be quite challenging since multiple conditions must be satisfied simultaneously. The hope is that a better understanding of the reorganizations will lead to a more clear and concise statemant of the necessary and sufficient conditions, it turn shedding some light on the creation of an effective algorithm.

Appendix A

MATLAB Code for level_periph

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  This function will determine if a set of Jordan blocks (all with
%  eigenvalues of the same modulus, assumed to be 1 WLOG) corresponds
%  to the peripheral spectrum of a nonnegative matrix.
%  The function takes as input a t x (m+2) matrix
%  A where m is the size of the largest Jordan block in the set with
%  eigenvalue 1 and t is the number of distinct eigenvalues in the set.
%  Each row of A correponds to a particular eigenvalue. We represent the
%  complex number exp(2*pi*k*i/n) by the ordered pair (k,n) and these
%  ordered pairs determine the first two columns of A. For j>2, the (i,j)
%  entry of A is the number of Jordan blocks of size (j-2)x(j-2)
%  with the ith eigenvalue.
%  For example, if m=4 and there are 8 Jordan blocks in the multiset
%  with eigenvalue exp(2*pi*i*k/n) where 3 are size 1, 4 size 2, and 1
%  size 3, then the corresponding row of A is [k n 3 4 1 0].
%  This program first checks to see that the multiset of eigenvalues can
%  be partitioned into complete sets of roots of unity. If so, a vector
%  R is formed that lists the sizes of the sets of roots of unity in the
%  partition in decreasing order.
```

```
%  Next, it is determined whether or not the set of blocks satisfies the
%  Extended Tam-Schneider condition. If so, a vector L is returned with
%  entries from {1,...,m} which indicates that the R(i)th roots of unity
%  should be placed in level L(i).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function[R,L] = level_periph(A)
A = sortrows(A); % Useful for check(A) and self_conj(A).
[t,m] = size(A);
m = m-2;
if ~check(A,t,m)  % Check to see that A is an appropriate input matrix.
    L = []; R = [];
    return
end
if ~self_conj(A,t) % Check to see if the multiset is self-conjugate.
    disp('Multiset does not correspond to peripheral spectrum of
          nonnegative matrix')
    disp('The multiset is not self-conjugate.')
    L = []; R = [];
    return
end
A(:,1) = A(:,1)./A(:,2);
[R,b] = partition(A,t,m);
if ~b
    disp('Multiset does not correspond to peripheral spectrum of a
```

```
        nonnegative matrix.')
    L = [];
    return
end
% Delete second column of A (no longer needed).
% Note that A is now tx(m+1) where the first column indicates e-vals and
% remaining columns the number of (m-1)x(m-1) blocks.
A(:,2) = [];
% Delete rows of A corresponding to eigenvalues only appearing in 1x1
% blocks. Also delete rows corresp to e-vals with argument greater than
% 2*pi.
rows = [];
for ii = 1:t
    if A(ii,3:m+1) == 0
        rows = [rows; ii];
    elseif A(ii,1) > 1/2
        rows = [rows; ii];
    end
end
A(rows,:) = [];
% Replace 2nd through last row of A with 'majorizing' vector.
t = length(A(:,1));
for jj = m:-1:2
    A(:,jj) = A(:,jj) + A(:,jj+1);
```

```
end
% initialize placement matrix P and level vector L
P = zeros(t,m);
L = [1];
done = 0;
[R,L] = place_roots(R,L,A,P,m);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function checks:
%   1. That all entries of A are nonnegative integers.
%   2. That eigenvalues are represented in lowest terms.
%   3. Each eigenvalue appears in only one row.
%   4. That, if m is the size of the largest Jordan block, then there is
%      an m-by-m block with eigenvalue 1.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function b = check(A,t,m)
for ii = 1:t
    for jj = 1:m+2
        if ~isequal(round(A(ii,jj)),A(ii,jj)) | A(ii,jj) < 0
            error('Entries of A must be nonnegative integers.');
        end
    end
    if gcd(A(ii,1),A(ii,2)) > 1 | A(ii,1)/A(ii,2) >= 1
        error('Eigenvalues must be represented in lowest terms.');
    end
```

```
    for jj = ii+1:t
        if A(ii,1:2) == A(jj,1:2)
           error('Each eigenvalue must be represented in only one row.');
        end
    end
end
if A(1,m+2) == 0 | A(1,1) ~= 0
    disp('Multiset does not correspond to the peripheral spectrum of a
         nonnegative matrix.')
    disp('There is not an m-by-m block with eigenvalue 1.')
    b = 0;
    return
end
b = 1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function checks to see that A corresponds to a self-conjugate
% collection of Jordan blocks
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function b = self_conj(A,t)
B = A;
for ii = 2:t
    B(ii,1) = A(ii,2)-A(ii,1);
end
B = sortrows(B);
```

```
if A == B
    b = 1;
else
    b = 0;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This function determines if the eigenvalues can be partitioned into
% complete sets of roots of unity.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [R,b] = partition(A,t,m)
% Add (m+3)rd column to A with total numbers of e-vals.
new_col = zeros(t,1);
for jj = 3:m+2
    new_col = new_col + (jj-2)*A(:,jj);
end
A = [A new_col];
% Determine sets in partition into complete sets of roots of unity.
R = [];
while A(1,m+3) > 0
    I = find(A(:,m+3)); n = max(A(I,2)); % nth roots
    for r = 0:n-1
        row = find(A(:,1)==r/n);
        if isempty(row) | A(row,m+3) == 0
            R = ('Eigenvalues do not partition into complete sets of
```

```
                  roots of unity.');
            b = 0;
            return
        else
            A(row,m+3) = A(row,m+3)-1;
        end
    end
    R = [R, n];
end
% Check that all eigenvalues were used (since above uses no ones left as
% stopping criterion).
nonzeros = find(A(:,m+3));
if length(nonzeros) ~= 0
    R = ('Eigenvalues do not partition into complete sets of roots of
          unity.');
    b = 0;
    return
end
b = 1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Recursive branch and bound algorithm for placing sets of roots in
% levels. The function searches a tree where nodes are pre-ordered.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [R,L] = place_roots(R,L,A,P,m)
```

```
if L(1) == 2  % All groupings fail
    disp('Multiset does not correspond to peripheral spectrum of
          nonnegative matrix.')
    L = 'No level assignments satisfy Extended Tam-Schneider Condition';
    return
end
root_num = length(L);
if L(root_num) > max(L(1:root_num-1))+1 % level was skipped so move on.
    [L,P] = bypass_children(L,P);
    [R,L] = place_roots(R,L,A,P,m);
end
if max(L) < m - length(R) + length(L) % cannot visit every level, move on
    [L,P] = bypass_children(L,P);
    [R,L] = place_roots(R,L,A,P,m);
end
% If the current set of roots is equal to previous, must stay at same
% level or move up a level.
if root_num > 1 & R(root_num)==R(root_num-1)&L(root_num)<L(root_num-1)
    L(root_num) = L(root_num-1);
end
% Update placement matrix, placing R(root_num) roots in level L(root_num).
for k = 0:R(root_num)-1
        row = find(A(:,1)==k/R(root_num));
        P(row,L(root_num)) = P(row,L(root_num))+1;
```

```
end
if majorize(A(:,2:m+1),P)
    if length(L) == length(R); % All roots placed successfully so done.
       done = 1;
       return
    else
        [L,P]=next_vertex(L,P,length(R)); % Not all roots placed yet, but
                                           % majorization OK.
    end
else
    for k = 0:R(root_num)-1  % 'Unplace' roots.
        row = find(A(:,1)==k/R(root_num));
        P(row,L(root_num)) = P(row,L(root_num))-1;
    end
    [L,P] = bypass_children(L,P); % Majorization failed, move to
                                  % next branch.
end
if isempty(P)
    P = calculate_P(A,L,R);
end
[R,L]=place_roots(R,L,A,P,m);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Updates L to next vertex in tree. If moving to a new branch,
%  sets P to empty.
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [L,P] = next_vertex(L,P,t)
m = length(P(1,:));
root_num = length(L);
if root_num < t
    L = [L 1];
else
    [L,P] = bypass_children(L,P);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Bypasses all children, grandchildren, etc. of input node L. If only
%  the last entry of L changes then P is left unchanged. Otherwise P is
%  set to empty to be calculated later using calculate_P subroutine.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [L,P] = bypass_children(L,P)
m = length(P(1,:));
root_num = length(L);

for jj = root_num:-1:1
    if L(jj) < m
        L(jj) = L(jj) + 1;
        L = L(1:jj);
        if jj == root_num
            return
```

```
        else
            P = [];
            return
        end
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  Calculates placement matrix P from A, L, and R.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function P = calculate_P(A,L,R)
root_num = length(L);
[t,m] = size(A);
m = m - 1;
P = zeros(t,m);
for r = 1:root_num-1
    for k = 0:R(r)-1
        row = find(A(:,1)==k/R(r));
        if isempty(row)
        else
            P(row,L(r)) = P(row,L(r)) + 1;
        end
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Checks to see if B = A(2:m+1) 'majorizes' P (ie Does row i of A
% 'majorize' row i of P for each i).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function b = majorize(B,P)
[n,m] = size(P);
P = sort(P')';
B = sort(B')';
for ii = 1:n
    for jj = 1:m-1
        for k = jj+1:m
            P(ii,jj) = P(ii,jj) + P(ii,k);
            B(ii,jj) = B(ii,jj) + B(ii,k);
        end
    end
end
if B >= P
    b = 1;
else
    b = 0;
end
```

Appendix B

MATLAB Code for J2NN

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  This function will output a nonnegative matrix with spectrum
%  corresponding to J (provided one exists).
%  This function calls the level_periph function to determine whether or
%  not a multiset of Jordan blocks corresponds to the peripheral spectrum
%  of a nonnegative matrix. See level_periph for details on input.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function[NN,R,L] = J2NN(J)
[R,L] = level_periph(J);
if ~isfloat(L)
    NN = 'There is no nonnegative matrix with Jordan form corresponding
          to J';
else
[t,m] = size(J); % t = number of distinct eigenvalues
m = m - 2;       % m = number of levels
% Compute RootsInLevel cell array.
% RootsInLevel{i} is a vector listing the root numbers in level i.
RootsInLevel = cell(1,m);
for ii = 1:m
```

```
    ind = find(L == ii);
    RootsInLevel{ii} = R(ind);
    LevelChar(ii) = length(ind);
end
% We now construct LJ, the cell array representing the matrix we will
% apply a series of similarity transformations to to get NN
% (LJ = "Leveled Jordan form").
% Diagonal blocks are represented by vectors consisting of unions of
% complete sets of roots of unity.
LJ = cell(m,m);
for s = 1:m
    LJ{s,s} = Z(RootsInLevel{s}(1));
    for ii = 2:length(RootsInLevel{s});
        LJ{s,s} = [LJ{s,s}, Z(RootsInLevel{s}(ii))];
    end
end
% Initialize off diagonal blocks to zero blocks.
for s = 1:m
    for c = 1:s-1
        LJ{s,c} = zeros(length(LJ{s,s}),length(LJ{c,c}));
    end
    for c = s+1:m
        LJ{s,c} = zeros(length(LJ{s,s}),length(LJ{c,c}));
    end
```

```
end

% Place ones in subdiagonal blocks connecting eigenvalues

for eigval = 1:t

    %if J(eigval,3:m+2)~=0 % All 1x1 blocks so skip to next eigenvalue.

        lambda = exp(2*pi*i*J(eigval,1)/J(eigval,2));

        for s = 1:m

            indcell{s} = find(LJ{s,s}==lambda);

            level_char(s) = length(indcell{s});

        end

        for Jblocksize = m:-1:2 % Start with largest block size.

            % for each block of above size,

            for BlockNumber = 1:J(eigval,Jblocksize + 2);

                % Find location of first lambda in seperate levels that have

                % not yet been connected.

                nb = find(level_char);

                for count = 1:Jblocksize - 1;

                    s = nb(count + 1);  % row s block

                    c = nb(count);      % column c block

                    LJ{s,c}(indcell{s}(1),indcell{c}(1)) = 1;

                    % Remove lambda that have just been connected from lists.

                    indcell{c} = indcell{c}(2:level_char(c));

                    level_char(c) = level_char(c) - 1;

                end

                indcell{nb(Jblocksize)} = indcell{nb(Jblocksize)}
```

```
                                        (2:level_char(nb(Jblocksize)));
                level_char(nb(Jblocksize)) =level_char(nb(Jblocksize))-1;
            end
        end
end
% celldisp(LJ)
% Set the (1,1) entry of each subdiagonal block to an appropriately large
% scalar.
for s = 2:m
    for c = 1:(s-1)
        LJ{s,c}(1,1)=m^2+max(R);
    end
end
% Now we assemble the matrix LJ from the cell array LJ.
for s = 1:m
    LJ{s,s}=diag(LJ{s,s});
end
LJ = cell2mat(LJ);
RootsInLevel = cell2mat(RootsInLevel);
% Apply a similarity transformation to the submatrix corresponding to
% lambda = 1.
ind = find(diag(LJ)==1);
[S,invS] = S1(LevelChar);
LJ(ind,ind) = S*LJ(ind,ind)*invS;
```

```
% Apply the final similarity transformation.
[S,invS] = simtrans(RootsInLevel);


NN = S*LJ*invS;
NN = real(NN);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 function Z = Z(n)
% This function will take the natural number n as input and output a
% vector Z of length n consisting of the nth roots of unity.
Z=[1];
for j=1:n-1
    Z=[Z exp(2*pi*i*j/n)];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [S,invS] = simtrans(R)
% Input: R, a vector. R_i is a natural number corresponding to
% exp(2*pi*i/R_i). These numbers will generate S.
% Output: blocks, a cell array containing the diagonal blocks of the
%                 transformation matrix.
%         S, a matrix which is the direct sum of the blocks.
t = length(R);
blocks{1} = subsim(R(1));
S = blocks{1};
```

```
invS = 1/R(1)*conj(blocks{1});
for ii=2:t
    blocks{ii} = subsim(R(ii));
    S = [S zeros(sum(R(1:ii-1)),R(ii));zeros(R(ii),sum(R(1:ii-1)))
        blocks{ii}];
    invS = [invS zeros(sum(R(1:ii-1)),R(ii));zeros(R(ii),sum(R(1:ii-1)))
             1/R(ii)*conj(blocks{ii})];
end
%%%%%%
function S = subsim(n)
alpha = exp(2*pi*i/n);
for jj = 1:n
    for kk = 1:n
        S(jj,kk) = alpha^((jj-1)*(kk-1));
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [S,invS] = S1(n)
% Creates block diagonal matrix S and its inverse, invS.
% The diagonal blocks satisfy: the first column is the all ones matrix,
%                              diagonal entries are ones, and
%                              remaining entries in row 1 are -1.
% invS is the inverse of S.
% Input: n, a vector determining the block sizes.
```

```
S = [1 -1*ones(1,n(1)-1); ones(n(1)-1,1) eye(n(1)-1)];
invS=1/n(1)*[ones(1,n(1));-1*ones(n(1)-1,1),-1*ones(n(1)-1)+n(1)*
        eye(n(1)-1)];
for ii = 2:length(n)
    next = [1 -1*ones(1,n(ii)-1); ones(n(ii)-1,1) eye(n(ii)-1)];
    S = blkdiag(S,next);
    next = 1/n(ii)*[ones(1,n(ii)); -1*ones(n(ii)-1,1),
                -1*ones(n(ii)-1)+n(ii)*eye(n(ii)-1)];
    invS = blkdiag(invS,next);
end
```

Bibliography

[1] A. Berman and R. J. Plemmons. Nonnegative Matrices in the Mathematical Sciences. Academic Press, New York, 1979.

[2] Sarah Carnochan Naqvi and Judith J. McDonald. The combinatorial structure of eventually nonnegative matrices. *The Electronic Journal of Linear Algebra.* 9, 255-269, 2002.

[3] Sarah Carnochan Naqvi and Judith J. McDonald. Eventually nonnegative matrices are similar to seminonnegative matrices. *Linear Algebra and its Applications.* 381, 245-258, 2004.

[4] Abed Elhashash and Daniel Szyld. Perron-Frobenius properties of general matrices. 5 November 2007.

[5] Daniel Hershkowitz. The combinatorial structure of generalized eigenspaces – from nonnegative matrices to general matrices. *Linear Algebra and its Applications,* 302/303, 173-191, 1999.

[6] Daniel Hershkowitz and Hans Schneider. On the existence of matrices with prescribed height and level characteristics. *Israel Journal of Mathematics,* 75, 105–117, 1991.

[7] Roger A. Horn and Charles R. Johnson. Matrix Analysis. Cambridge University Press, Cambridge, 1999.

[8] Neil C Jones and Pavel A. Pevzner. *An Introduction to Bioinformatics Algorithms.* Massachusetts Institute of Technology, Cambridge, 2004.

[9] Judith J. McDonald. The Peripheral Spectrum of a Nonnegative Matrix. *Linear Algebra and its Applications,*363, 217-235, 2003.

[10] J. J. McDonald and B. G. Zaslavsky. A characterization of Jordan canonical forms which are similar to eventually nonnegative matrices with the properties of nonnegative matrices. *Linear Algebra and its Applications*, 372, 253285, 2003.

[11] D. Richman and H. Schneider. On the singular graph and the Weyr characteristic of an M–matrix. *Aequationes Math.* 17, 208-234, 1978.

[12] U. G. Rothblum. Algebraic Eigenspaces of Nonnegative Matrices. *Linear Algebra and its Applications*, 12, 281-292, 1975.

[13] Hans Schneider. The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: a survey. *Linear Algebra and its Applications,* 84, 161-189, 1986.

[14] Bit–Shun Tam. A cone–theoretic approach to the spectral theory of positive linear operators: the finite dimensional case. *Taiwanese Journal of Mathematics,* 5, no. 2, 207-277, 2001.

[15] Bit–Shun Tam and Hans Schneider. On the core of a cone-preserving map. *Trans. Amer. Math. Soc.,* 343, 479-524, 1994.

[16] B. G. Zaslavsky and Bit–Shun Tam. On the Jordan form of an irreducible matrix

with eventually non-negative powers. *Linear Algebra and its Applications,* 302-303, 303-330, 1999.