GÖDELIAN PLATONISM

By

TEPPEI HAYASHI

A thesis submitted in partial fulfillment of
the requirements for the degree of

Master of Arts in Philosophy

WASHINGTON STATE UNIVERSITY
Department of Philosophy

MAY 2009

To the Faculty of Washington State University:

The members of the Comittee appointed to examine the thesis of TEPPEI HAYASHI find it satisfactory and recommend that it be accepted.

David Shier, Ph.D., Chair

Joseph Keim Campbell, Ph.D.

Harry Silverstein, Ph.D.

ACKNOWLEDGEMENT

First, I thank my advisor David Shier for his generous support. Without his suggestions and encouragement, I could not have completed this thesis. I also thank Joe Keim Campbell and Harry Silverstein who accepted the position of the committee member with good grace. Last not but least, I thank all members of the Department of Philosophy at Washington State University. My philosophical development has been significantly due to conversations with them, whether these took place in or outside class.

GÖDELIAN PLATONISM

Abstract

By Teppei Hayashi, MA
Washington State University
May 2009

Chair: David Shier

The primary objective of this thesis is to clarify the philosophical view about mathematics of the great 20th century logician/mathematician Kurt Gödel.

Gödel's philosophical view of mathematics was well known during his lifetime. His view of mathematics is usually called *mathematical realism* or *mathematical Platonism* according to which mathematics is about objective existence. Gödel's view of mathematics — *Gödelian Platonism* — had long been often regarded by many as naïve and amateurish. However, the publication of Gödel's *Collected Works* (1986-2003) sheds new light on his thought.

In this paper, based on new insights and documents which have become available through the publication of *Collected Works*, I will try to show that Gödelian Platonism is not implausible, as some philosophers still think. To show this, I will concentrate on two principal characteristics of Gödelian Platonism: conceptual realism and mathematical intuition. In the first chapter, I will review how Gödelian Platonism has been interpreted. This will give readers a broad idea about Gödelian Platonism and make clear the points which I will discuss in greater details in the later chapters. In the second chapter, I will deal with an important aspect of Gödelian Platonism: conceptual realism. Gödelian Platonism had long

been thought as a kind of realism *simpliciter*, that is, the view that admits the objective existence of abstract objects like numbers, sets, and so forth. However, besides having believed in the existence of abstract objects themselves, Gödel seems to have also believed in the existence of concepts of abstract objects. This conception of Gödelian Platonism sheds lights on how we should understand Gödel's philosophy of mathematics. In the third chapter, I will discuss another important aspects of Gödelian Platonism: intuition. This might be the most notorious aspect of Gödelian Platonism because it had been thought of as a kind of perception through which we can directly access mathematical objects. I will try to demystify this aspect of Gödelian Platonism.

TABLE OF CONTENT

Page

ACKNOWLEDGEMENT....iii

ABSTRACT....iv

CHAPTER

INTRODUCTION....1

1. OVERVIEW OF GÖDEL'S PHILOSOPHICAL DEVELOPMENT....3

2. CONCEPTS IN GÖDELIAN PLATONISM....35

3. MATHEMATICAL INTUITION....61

CONCLUDING REMARKS....85

BIBLIOGRAPHY....88

I dedicate this thesis to my parents

who provided both emotional and financial support

and to my wife

who always tolerates my absent-mindedness

Introduction

Gödel's Platonism was well-known during his lifetime; or rather notorious. It was regarded by many as very naïve and even amateurish. Chihara's following evaluation exemplifies this dismissive attitude.

> . . . [E]ven supporters of the Gödelian view must admit that there are features of the view that make it difficult to accept. The mathematician is pictured as theorizing objects do not exist in physical space. This makes it appear that mathematics is a very speculative undertaking, not very different from traditional metaphysics. A mysterious faculty is postulated to explain how we can have knowledge of these objects. Gödel's appeal to mathematical perceptions to justify his belief in sets is strikingly similar to the appeal to mystical experiences that some philosophers have made to justify their belief in God. Mathematics begins to look like a kind of theology.[1]

Now the situation has changed. Since the publication of the volume III of Gödel's *Collected Works*, Gödelian Platonism has been regarded as less naive and amateurish as Chihara depicts. In fact, Gödelian Platonism is the fruits of deliberate thoughts over the years, not a half-baked idea. Gödel surely thought of mathematical objects as ones which "do not exist in physical space" and even believed that such objects really exist independently of us. He also thought that a faculty called "intuition" plays an important role in communicating with mathematical objects. However, these thoughts are quite different from

[1] [Chihara 1990, p. 21].

those which Chihara attributes to Gödel's. Although there is something "metaphysical" in Gödelian Platonism, it is not so mysterious or mystical as Chihara contends.

In this paper, I will argue that Gödelian Platonism is not implausible as some philosophers still think. To show the plausibility of Gödelian Platonism, I will concentrate on two principal aspects of Gödel's argument for his Platonism: conceptual realism and intuition. In Chapter 1, I will review how Gödelian Platonism has been interpreted, or I would venture to say, how it has been misinterpreted. This will give readers a broad idea of Gödelian Platonism and make clear the points which I will discuss in greater detail in the later chapters. In Chapter 2, I will deal with an important aspect of Gödelian Platonism: conceptual realism. Gödelian Platonism had long been thought as a kind of realism simpliciter, that is, the view that admits the existence of abstract objects like numbers, sets, and so forth. However, besides having believed in the existence of abstract objects themselves, Gödel seems to have also believed in the existence of concepts of abstract objects. This conception of Gödelian Platonism sheds lights on how we should understand Gödel's philosophy of mathematics. In Chapter 3, I will discuss another important aspect of Gödelian Platonism: intuition. This aspect of Gödelian Platonism might be the most notorious one because, as typically shown in the Chihara's writing quoted above, it had been thought as a kind of perception through which we know mathematical objects. If such an interpretation were adopted, it would be no wonder that Gödelian Platonism is taken to be mystical as a whole. However, with the conceptual realist aspect of Gödelian Platonism taken into account, a more plausible interpretation of what Gödel says about intuition can be possible.

Chapter 1
Overview of Gödel's Philosophical Development

In an unsent response to Burke D. Grandjean's questionnaire asking about Gödel's "intellectual and biographic" background,[1] Gödel said that his Platonist or realist view toward mathematics had been his position since 1925 when he was just a teenager. Since then, he seems to have kept his Platonist position. However, it does not mean that the content of his position remained the same through out his life. In this chapter, I will trace Gödel's struggle to make his Platonist position more plausible. In doing so, I will also give some responses to the criticism against Gödelian Platonism, and make clear the points to be developed in the later chapters.

1 "The present situation in the foundations of mathematics" (1933)

The first time Gödel expressed his Platonist view in public is, as far as we can know from the published documents, at a meeting of the Mathematical Association of America, held on 29-30 December, 1933. In the meeting, Gödel gave a lecture entitled "The present situation in the foundations of mathematics," in which he expounded his Platonist view, some aspects of which were repeatedly stated in his later writings.

Gödel starts his lecture by dividing the problem concerning the foundations of mathematics into two parts: to state axioms and rules of inference as accurately and efficiently

[1] [Gödel 2003, pp. 446-450].

as possible and to justify these axioms and rules.[2] As to the former part of the problem, Gödel says that there has been a "perfectly satisfactory" answer which was given in "the so-called formalization of mathematics."[3] After making some remarks regarding this part of the problem, Gödel moves to the second part, the examination of which comprises the main part of his lecture.

In contrast to the first problem for which there exists the "perfectly satisfactory" solution, the situation as to justifying axioms and rules of inference remains "extremely unsatisfactory."[4] To see why, we should examine what Gödel proposes as the possible solutions to the problem and how he evaluates these solutions.

First, Gödel says that there is nothing problematic in justifying axioms and rules of inference if we regard mathematics as "a mere game of symbols" which do not have meaning at all.[5] However, once we try to assign meanings to these symbols, we confront serious difficulties. Gödel names three difficulties.

The first difficulty is concerning "the non-constructive notion of existence." If we accept this kind of the notion of existence, we are always allowed to state propositions like

---

[2]This way of founding mathematics is called *formalism*. It was initiated by David Hilbert in the late nineteenth century and partially realized in his *Grundlagen der Geometrie*.

[3][Gödel 1933, p. 45].

[4][Gödel 1933, p. 49].

[5]The characterization of formalism as "a mere game of symbols" is, though a popular one, far too oversimplified. Actually, Hilbert himself criticizes such a characterization. He says that "this formula game is carried out according to certain definite rules, in which the *technique of our thinking* is expressed" ([Hilbert 1928, p. 475]; italics in the original). Clearly enough, this game of symbols is far from devoid of meaning.

"there exist some objects which have a certain property P" even if we do not know what these objects really are or how we can get to know them.[6] That is, the non-constructive notion of existence, unlike the constructive one, allows us to assume that some objects with or without some properties exists independently of us. This notion of existence has, however, provoked criticism that it is strange to assume that one can talk about something even when he or she does not know what it is at all.[7]

The second difficulty is about impredicative definitions. Gödel argues that if we have to do mathematics without any impredicative definitions, that is, without the assumption that there is the totality of mathematics in terms of which mathematical objects are defined, we are to lose considerable parts of mathematics including the theory of real numbers.[8] In other words, we can capture the meanings of mathematics only partially with the prohibition of the impredicative definitions. If we want to keep mathematics intact, we have to assume the totality of mathematics which exists without being known to us and as a result, independently of us.

The last difficulty is that of the axiom of choice. Although Gödel says almost nothing

---

[6]Here, we suppose the law of excluded-middle. In fact, supposing the law of excluded-middle also presupposes the non-constructive notion of existence.

[7]One might recall the following characterization of mathematics by Russell: "[M]athematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true" ([Russell 1957, p. 71]).

[8]Historically, the first person who explicitly objected to the use of the impredicative definitions was perhaps Poincaré. In Poincaré 1906, he says that "one cannot define [a certain set] E by E itself" ("on ne peut pas définir E par l'ensemble E lui-même," [Poincaré 1906, p. 206]) and such a definition of E contains a vicious circle ("un circle vicieux").

about this aspect of the difficulties, we can easily spot why the axiom should be regarded as problematic. Very roughly speaking, what the axiom of choice allows to assume is that there exists a choice function for any sets. With a certain choice function, you can select exactly one element at once from each member of infinite sets. In assuming such a choice function, we need not know what such a function exactly is or how we can construct it. The situation is completely similar to the two difficulties mentioned above. If we want to use the axiom of choice, we have to assume that *all* the choice functions exist "somewhere" independently of us.

From the analysis of these difficulties, Gödel draws the following conclusion.

> The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent.[9]

Solomon Feferman says that this conclusion is "most surprising" because the conclusion seems inconsistent with what Gödel said in the response to Grandjean's questionnaire which asked Gödel's intellectual background. Gödel said in his response that his Platonist or realist view toward mathematics had been his position since 1925 when he was just a teenager.[10] In short, Feferman thinks that the above conclusion shows that Gödel rejected Platonism at that time when he delivered this lecture, even though Feferman admits that

[9] [Gödel 1933, p. 50].

[10] [Gödel 2003, pp. 446-450].

Gödel advocated the Platonist conception of mathematics later in his career.[11] However, does the conclusion really imply his rejection of Platonism?

First, it should be noted that Gödel did not want to interpret mathematics in the formalist way at all for he devotes almost all the latter half of this lecture to interpret mathematics as meaningful. Second, he does not seem to take the position in which the non-constructive notion of existence, impredicative definitions, and the axiom of choice are to be discarded in doing mathematics. Certainly, it is true that Gödel examines the constructivist way of doing mathematics in some detail. However, despite admitting that in the future classical arithmetic and analysis could be built in the constructivist way, Gödel ultimately judges the constructivist approach as not so satisfactory. As to the other two, that is, impredicative definitions and the axiom of choice, Gödel did not give any clue for the argument that these are to be discarded. Then what remains as an alternative for Gödel is Platonism only, even if he was not completely satisfied with that position.

2 "Russell's mathematical logic" (1944)

In 1942, Paul Arthur Schilpp invited Gödel to contribute a paper for the Russell volume of *The Library of the Living Philosophers* series each volume of which contains critical papers about a certain philosopher by prominent thinkers who are not necessarily philosophers and the philosopher's replies. Gödel accepted and wrote a critical paper titled "Russell's mathematical logic." Although the paper is about Russell's thought, it also exhibits Gödel's own because of its critical nature. In other words, Gödel expresses his own thought in this

[11] See his introductory note in [Gödel 1995, pp. 39-40].

paper through the interpretation and critique of Russell's thought. Three aspects of Gödel's thought expressed in the Russell paper particularly draw our attention — impredicative definitions, Russell's defense of realistic view toward mathematics, and Gödel's own conception of "class" and "concept." Because the third one will be dealt with in the next chapter, I will examine only the first two in this section.

Impredicative definitions are, as I briefly mentioned in the previous section, those done by referring to a totality which is supposed to be comprised of elements defined in terms of that totality. For example, the Russell's famous paradox, which can be formally expressed as $\{x\colon x \notin x\}$, involves an impredicative definition because the definiens (the left side of the formula) contains (actually, in this instance, is equals) the definiendum (the right side). In other words, the set x is defined in terms of itself. Clearly seen, there seems a circularity in this way of definitions. Poincaré and Russell regard such a circularity as the cause of paradoxes and call it *vicious circle*. A principle which prohibits the vicious circles is *the vicious-circle principle*.

Gödel argues that this principle, especially as it appears in Russell and Whitehead's *Principia Mathematica*, can be formulated as the principle that "no totality can contain members definable only in terms of this totality, or members involving or presupposing this totality."[12] Gödel then asserts that there in fact exist three different kinds of the vicious circle principles according to the phrases "definable only in terms of," "involving," and "presupposing" in the above formulation.[13] It is the first kind with which Gödel primarily

[12] [Gödel 1944, p. 125].

[13] [Gödel 1944, p. 127]. Although Gödel does not explicitly refer to the corresponding parts in *Principia* to each phrase in the Gödel's formulation of the vicious circle principle, we can figure out with relative ease

concerns himself.

While Gödel says that "the second and the third [kinds of the vicious circle principles are] much more plausible than the first,"[14] he criticizes the first kind as follows.

> . . . [O]nly this one [the first kind of the principles] makes impredicative definitions impossible and thereby destroys the derivation of mathematics from logic . . . and a good deal of modern mathematics itself.[15]

Here, the vicious circle principle implies the impossibility of impredicative definitions and then that of "a good deal of modern mathematics." This is exactly the similar situation to that which was discussed in [Gödel 1933].[16] However, unlike his seemingly awkward position in 1933, Gödel takes a significant step toward the defense of his Platonist view this time.

First, Gödel argues that the formalization of classical mathematics of which Dedekind and Frege are supposed to be the "fathers" actually uses impredicative definitions.[17] More-

---

where those phrases come from. The original phrases corresponding to "definable in terms of," "involving," and "presupposing" are the following: "provided a certain collection had a total, it would have members only definable in terms of that total," "[w]hatever involves *all* of a collection must not be one of the collection," and "if we suppose the set to have a total, it will contain members which presuppose this total" (all of these can be found in [Russell and Whitehead 1925, p. 37]).

[14] [Gödel 1944, p. 127].

[15] [Gödel 1944, p. 127].

[16] See p. 4 of this paper.

[17] Among the uses of impredicative definitions in classical mathematics, ones in analysis are perhaps most paradigmatic. For instance, the definition of least upper bound is one of such definitions. As to the definition of least upper bound as an impredicative one, see [Kleene 1952, pp. 42-43].

over, he points out that the system of Russell himself, that is, the system of *Principia Mathematica* does not meet the vicious circle principle in the first form mentioned above because the system contains the axiom of reducibility which presupposes without enough argument or justification that any impredicative definition has its predicative version.[18] Then, Gödel suggests that "this [is to be considered] rather as a proof that the vicious circle principle is false than classical mathematics is false."[19] The vicious circle principle in the first form is valid or, in the other words, impredicative definitions are not allowable in mathematics "only if one takes the constructivistic (or nominalistic) standpoint toward the objects of logic and mathematics."[20] Unlike constructivists or nominalists, Gödel actually thinks that "objects of logic and mathematics" really exist.[21]

One of the noticeable characteristics of how Gödel defended his Platonist view of mathematics throughout his lifetime is in what Chihara calls "the equi-supportive claim."[22] According to such a claim, we should admit the reality of mathematical objects if we admit

[18][Russell and Whitehead 1925, pp. 55-59]. There, Russell argues that "after some finite number of steps, we shall be able to get from non-predicative function to a formally equivalent predicative function" because "[t]he axiom of reducibility is equivalent to the assumption that any combination or disjunction of predicates is equivalent to a single predicate"' (pp. 58-59). In the note attached to the phrase "combination or disjunction of predicates," Russell requires that the number of predicates is finite. However, as in the case of the definition of least upper bound, "the number of predicates" can be infinite. Therefore, it turns out that, even with the axiom of reducibility, some important definitions cannot be done.

[19][Gödel 1944, p. 127].

[20][Gödel 1944, p. 128].

[21][Gödel 1944, p. 128].

[22][Chihara 1982, p. 212].

that of physical objects. Although this claim has already appeared as a quotation from Russell in the very first part of the paper,[23] it is almost in the middle part that Gödel explicitly commits himself to this claim for the first time in this paper.

> It seems to me that the assumption of such objects [objects of logic and mathematics] is quite as legitimate as the assumption of physical objects and there is quite as much reason to believe in their existence.[24]

The explanation Gödel gives for this claim has been a source of controversies concerning Gödelian Platonism, especially those about mathematical intuition.

> They [objects of logic and mathematics] are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about "data," i.e., in the latter case the actually occurring sense perceptions.[25]

Some interpreters infer from the analogy between mathematical objects and physical bodies that Gödel presupposes a kind of "mathematical intuition" which enables us to perceive

---

[23][Gödel 1944, p. 120]. The quotation from Russell is: "Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features" ([Russell 1920, p. 169]; cited in [Gödel 1944, p. 120]).

[24][Gödel 1944, p. 128].

[25][Gödel 1944, p. 128].

mathematical objects just as sense perceptions enable us to perceive physical bodies.[26] The analogies such interpreters infer from what Gödel says above can be schematized by using congruence expression as follows.

> mathematical object : mathematical intuition : a system of mathematics =
> physical bodies : sense perception : a theory of sense perception

However, as a little closer look reveals, it is not "mathematical intuition," but "data" which is paralleled with "sense perceptions." The term "mathematical intuition" does not even appear in the paragraph. To clear this point, we should refer to another significant characteristic in Gödelian Platonism which also has its root in the thought of Russell. I would like to call it the "consequentialist argument."

The consequentialist argument, in short, asserts that if assuming the existence of "something" in a theory enables new discoveries or developments in that theory, we are allowed to assume that such a "something" exists.

> I think that . . . this view has been largely justified by subsequent developments [of some theory], and it is to be expected that it will be still more so in the future. It has turned out that . . . the solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic, i.e., the domain of the kind of elementary indisputable evidence

[26]Charles S. Chihara and Penelope Maddy are among the major figures who interpret the quotation in the above way. However, while Chihara criticizes Gödel for presupposing "a mysterious faculty" ([Chihara 1990, p. 23]) such as mathematical intuition, Maddy tries to defend Gödel's argument by "naturalizing" it ([Maddy 1997, pp. 89-94]). We will return to this point in the third chapter.

that may be most fittingly compared with sense perception.[27]

This consequentialist argument suggests that what Gödel wanted to express by using the analogy between mathematics and "a theory of sense perception" could be paraphrased as follows. First, in constructing a theory of sense perception, we have to organize sense perceptions which are not well-organized by themselves. For this purpose, we need to assume the existence of physical objects which are supposed to be the sources of sense perceptions and properties accompanying with such objects. By assuming the existence of physical objects (and their properties) and thinking within the domain of such objects, we can consequently predict or explain some phenomena concerning sense perception which are not predictable or explainable as far as we think only about sense perception. In other words, such predictions or explanations about sense perceptions, that is, propositions about them, "transcend" sense perceptions themselves and therefore cannot be reduced to them.

The same as in the sense perception case can be said for the case of mathematics. Let us suppose here that there is a conjecture C in arithmetic which is supposed to hold for all natural numbers and suppose also that any counter-example has not been found so far. However, the fact that any counter-example has not been found, that is, the fact that C actually holds for natural numbers which has been tested so far does not prove C at all. In other words, mere "data" which are, in this case, that C holds for any natural numbers tried so far cannot establish the status of C as theorem. Now, suppose that we proved C by using some axioms or some concepts which are not of arithmetic. In this case, as in that of sense perception, it follows that we need to assume the existence of objects which transcend

[27][Gödel 1944, p. 121].

"data" to prove C.

As seen above, two important aspects of Gödelian Platonism are explained in Gödel's Russell paper: one is the analogy between mathematics and physical sciences, and the other is the consequentialist argument. Both of them appear again and again in the later papers of Gödel and we will see in the following how these aspects have developed.

3 "What is Cantor's continuum problem?" (1947)

In 1945, Lester R. Ford, the editor of the *American mathematical monthly*, asked Gödel to write a paper about the continuum problem "in as simple, elementary and popular a way as [possible]."[28] In 1947, over a year after Ford's request, Gödel finally turned in his paper to the new editor, C. V. Newsome. His paper was quite densely written and consequently not "in as simple, elementary and popular a way as" expected.

The paper is divided in to four sections: 1) an explanation about the concept of cardinal number, 2) a survey of the results which had been obtained until that time, 3) a philosophical reflection on the foundations of set theory, and 4) a suggestion for further investigations about the continuum problem.[29] Among these sections, we will concentrate

[28][Gödel 1990, p. 159].

[29]In fact, in addition to these sections, the revised version of [Gödel 1947] which was written in 1964 has two additional sections. In the first of these additional sections titled "Postscript" which was actually written in 1966, Gödel mentions the independence of the continuum hypothesis from the ordinary axioms of set theory (which are usually called "ZFC axioms") which was proved by Paul Cohen after [Gödel 1947] was published. In the second additional section titled "Supplement to the second edition," Gödel deploys a dense argument about his Platonism and mathematical intuition. We will extensively devote the last section

on the third one in which Gödel expresses his Platonist conception of set theory.

Before going into the main point, it would be convenient for later purposes to briefly summarize what the continuum problem is and the situation of the problem at the time when Gödel wrote the paper. Gödel says, as to the former point, that the problem is "to find out which one of the $\aleph$'s is the number of points on a straight line,"[30] i.e., what is the cardinality of the real numbers is.[31] More roughly, it is about how many points there are in a straight line. Cantor conjectured that there are $\aleph_1$ points in a straight line.[32] This conjecture is called *the continuum hypothesis*. As to the latter point, although the consistency of Cantor's hypothesis with the axioms of set theory had already been proved by Gödel himself in 1938, it still remained unknown at that time whether the negation of the hypothesis is consistent with the axioms, or in other words, whether the problem is decidable in set theory. In short, the problem was completely unsettled in 1947. Gödel says as to this unsatisfactory situation of the problem as follows.

> This scarcity of results, even as to the most fundamental questions in this

---

of Chapter 3 for examining Gödel's argument in the supplement. Thus, in this section, we will intentionally ignore these additional sections.

[30][Gödel 1947, p. 177].

[31]$\aleph$ or a cardinality of a set is a measure of the size of that set. When there is a one-to-one correspondence between two sets, these sets are said to have the same cardinality. By the diagonal argument of Cantor, it can be shown that a cardinality of the natural numbers and that of the real numbers are not equal. The cardinality of the natural numbers is usually expressed as $\aleph_0$. $\aleph_1$ is the smallest cardinality which is greater than that of the natural number.

[32]From what is said in the above footnote, the conjecture of Cantor can be paraphrased as "there is no intermediate cardinality between the cardinality of the natural numbers and that of the real numbers."

> field, may be due to some extent to purely mathematical difficulties; it seems, however, that there are also deeper reasons behind it and that a complete solution of these problems can be obtained only by more profound analysis . . . of the meanings of the terms occurring in them (such as "sets," "one-to-one correspondence," etc) and of the axioms underlying their use.[33]

Gödel himself thought that the problem was undecidable with the current set of axioms of set theory even quite a while before Cohen actually proved its undecidability in 1963. However, for Gödel, the proof of the undecidability of the problem is not the final solution to the problem at all.

> It is to be noted, however, that, even if one should succeed in proving its undemonstrability as well, this would . . . by no means settle the question definitively. Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning . . . could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality . . .[34]

Here, at this point, his Platonist conception of mathematics provide a motivation to the further investigation of the continuum problem. For Gödel or those who have the

---

[33][Gödel 1947, p. 179].

[34][Gödel 1947, p. 181].

same Platonist conception of mathematics as Gödel, the continuum problem must be either true or false (of course not both) in the "mathematical universe" even if we are still not in position to know the answer of the problem. But how will we be able to come up with the answer? As implied in the last sentence of the above quotation, if our axioms do not suffice to capture the "reality," we should try to find new axioms which will provide us with "a complete description of this reality." Such axioms which Gödel proposes as the key to the solution of the continuum problem are so-called "large cardinal axioms" based on his iterative conception of set.

According to the iterative conception of set,

> a set is anything obtainable from the integer (or some other well-defined objects) by iterated application of the operation "set of," and not something obtained by dividing the totality of all existing things into two categories . . .[35]

Faced with the above explanation, one might naturally think that Gödel contradicts himself by adopting the iterative conception of set because he rejected such a constructivist way of doing mathematics in his "Russell's mathematical logic."[36] However, the operation "set of" can be iterated "transfinitely" and then this seemingly "constructivist" way of the iterative conception of set does not exclude the possibility of thinking the totality of sets. Actually, in the footnote attached to the phrase "iterated application" appeared in the above quotation, Gödel notes as follows.

---

[35] [Gödel 1947, p. 180].

[36] See the previous section.

> This phrase [iterated application] is to be understood as to include also transfinite iteration, the totality of sets obtained by finite iteration forming again a set and a basis for a further application of the operation of "set of."[37]

This means that Gödel just thinks of a somewhat different kind of totality from that which is thought of by those who regard a set as "something obtained by dividing the totality of all existing things into two categories." In either case, the totality of sets has its own reality[38] and contains something which cannot be attained with the constructivist way. For example, large cardinals are unattainable in the constructivist way.

"Large cardinal axioms" are, simply put, those which assert the existence of large cardinals. In other words, the axioms allow us to iterate the operation of "set of" infinitely. For example, starting with the null set which has nothing as its member, we can "construct" the set of all natural numbers which has, needless to say, infinite members and whose cardinality is $\aleph_0$. Then, repeating this process again and again, we can attain sets whose cardinalities are much greater than $\aleph_0$. Gödel thinks that by adding such large cardinal axioms to the ordinary axioms of set theory, the continuum problem could be settled.[39] However, even if large cardinal axioms are of great help to solve the continuum problem, how can it be

---

[37][Gödel 1947, p. 180].

[38]Gödel's confidence in the existence of such a totality is partly expressed in the following statement: "This concept of set . . . has never led to any antinomy whatsoever; that is, the perfectly 'naïve' and uncritical working with this concept of sets has so far proved completely self-consistent" ([Gödel 1947, p. 180]).

[39]Actually, in 1945 when Gödel wrote the first version of "Cantor's continuum problem," he thought that "there is little hope of solving it [the continuum problem] by means of those axioms of infinity [i.e., large cardinal axioms] which can be set up on the basis of principles known today" ([Gödel 1947, p. 182]). However, in the revised version published in 1964, he says that "from an axiom . . . the negation of Cantor's

justified to add new axioms like large cardinal axioms to an existing axiom system? Gödel replies to this question by appealing the "success" which the introduction of new axioms brings about.

> . . . [E]ven disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success," that is, its fruitfulness in consequences and in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proof by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs.[40]

A remarkable point in the above quotation is the use of the modifier "inductively." Contrary to common belief that mathematics is preeminently deductive, Gödel maintains that mathematics has also inductive aspects which could play an creative role in mathematics. In this respect, mathematics and empirical sciences are not so different. Actually, following the above quotation, Gödel brings up again the analogy between mathematics and physical science.

> There might exist axioms so abundant in their verifiable consequences, shed-

conjecture could perhaps be derived. I am thinking of an axiom which . . . would state some maximum property of the system of all sets" ([Gödel 1964, pp. 262-263]). Such an axiom which Gödel thinks of is clearly of large cardinal axioms.

[40] [Gödel 1947, p. 182].

> ding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems . . . that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well-established physical theory.[41]

As seen in the previous section, the Gödel's main strategy in justifying his Platonism is to appeal to the analogy between mathematics and physical sciences. This time, while the analogy in "Russell's mathematical logic" which is used in arguing the ontological aspect of mathematics, it is expanded for its methodological aspect.In this 1947 paper, Gödel has argued that taking the Platonist standpoint towards mathematics enables one to meaningfully pursue the investigation of the continuum problem and, moreover, presents a path towards the solution of the problem. However, Gödel almost exclusively argues about the benefits of the results which could be obtained by admitting Platonism, not about the validity of Platonism by itself. To be able to justify his Platonism *per se*, Gödel needed more time to contemplate.

4 "Some basic theorems on the foundations of mathematics and their implications" (1951)

In 1951, Gödel delivered a lecture entitled "Some basic theorems on the foundations of mathematics and their implications" at a meeting of the American Mathematical Society. The most notable feature of this lecture is the use of his incompleteness theorem to defend

[41] [Gödel 1947, pp. 182-183].

his Platonist view of mathematics.

Gödel, in the first part of the lecture, says that "[t]he metamathematical results I have in mind are all centered around . . . one basic fact, which might be called the incompletability or inexhaustibility of mathematics."[42] This "incompletability or inexhaustibility of mathematics" becomes explicit in the axiomatization of set theory and the implication of the incompleteness theorem. As to the former case, Gödel succinctly summarizes the reason why mathematics is incompletable by saying that "the very formulation of the axioms up to certain stage gives rise to the next axiom."[43] As to the latter, Gödel explains the incompletability by saying that, as far as axioms and rules of inference of a certain system are consistent, that system cannot contain all of mathematics.

> [The second incompleteness theorem] makes it impossible that someone should set up a certain well-defined system of axioms and rules [of inference] and consistently make the following assertion about it: All of these axioms and rules I perceive (with mathematical certitude) to be correct, and moreover I believe that they contain all of mathematics.[44]

From this implication of the second incompleteness theorem, Gödel draws the following conclusion: "Either . . . the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable

[42][Gödel 1951, p. 305].

[43][Gödel 1951, p. 307]. Gödel must have had large cardinal axioms in mind. As for large cardinal axioms, see pp. 16-17 of this paper.

[44][Gödel 1951, p. 309]; the original is in italics.

diophantine problems of the type specified."[45] Then, Gödel argues that the Platonist view concerning mathematics follows from the second alternative because if mathematics were, as constructivists assert, a creation of the human mind, it would not be the case that there exist absolutely unsolvable problems. To ascertain this implication from the second alternative, Gödel takes two possible objection and tries to respond to them.

The first objection is that "the constructor need not necessarily know *every* property of what he [or she] constructs."[46] Hence, it is no surprising at all that there exist unsolvable problems in mathematics which is supposed to be our creation. Gödel promptly rejects this objection as "very poor"[47] because we cannot create anything out of nothing and therefore what we create has inevitably contains what we cannot create, that is, some objective materials which exist independently of us. Thus, even if we admit that mathematics is partially created by us, there still remain something objective in mathematics.

Unfortunately, this Gödel's reply is not so convincing. Gödel, in the first place, asserted the objective existence of mathematical objects. However, in the above objection, it is what comprises mathematical objects that is asserted as having objective existence. If we could argue in the above way to assert the objective existence of mathematical objects, we could also conclude that what we create exists objectively, that is, independently of us. This is absurd.

The second objection goes as follows. Let us think of a proposition about all integers

[45] [Gödel 1951, p. 310]; the original is in italics. Gödel does not exclude the possibility that both alternatives hold.

[46] [Gödel 1951, p. 312]; italics in the original.

[47] [Gödel 1951, p. 312].

and suppose that the meaning of a proposition consists in its proof. Then, if the proposition happens to be undecidable, both the proposition and its negation are regarded as meaningless because there is no proof of them. Gödel replies to this objection by rejecting the identification of the meaning of a proposition with its proof.

What is worth noting in his reply to the second objection is that Gödel tries to justify the inductive method to verify whether some proposition is true or not by appealing the analogy between mathematics and physical sciences. Admitting that "every mathematician has an inborn abhorrence"[48] of the inductive way to verify a proposition, Gödel says that this mathematicians' abhorrence is "due to the very prejudice that mathematical objects somehow have no real existence."[49] Although this defense, or justification, of inductive methods in mathematics does not do much good to justify the objective existence of mathematical objects because the methods rather presuppose it, the main characteristic of Gödelian Pltonism is apparent here.

Gödel, besides the above counter-arguments, proposes the reasons why we should reject the "creationist" view and take the Platonist one towards mathematics. He asserts that such reasons are provided by the development of the foundations of mathematics. Firstly, for those who think that mathematics is a product of the human mind, the fact that there exists unsolvable problems in mathematics results from the lack of exactness in recognizing the product. However, despite the fact that we now have extreme precision thanks to the development of the foundations of mathematics, we still have plenty of unsolvable problems in mathematics. This shows, Gödel argues, that mathematics cannot be the product of the

[48][Gödel 1951, p. 313].

[49][Gödel 1951, p. 313].

human mind. Hence, mathematics has an objective existence.

Secondly, if mathematics were our creation, we could have freedom at least in part in doing mathematics. Yet, the fact is completely contrary. Even if we creates axioms, theorems which are deduced from these axioms are not at our disposal at all. This also shows that mathematics cannot be created by the human mind.

Lastly, if mathematics is a creation of the human mind, so are integers and sets of integers. Because integers and sets of integers are two different objects, that we created integers does not necessarily entail that we should also create sets of integers. However, we sometimes need sets of integers to prove some propositions about integers. This situation is "very strange."[50]

As George Boolos points out in the introductory note to this lecture,[51] even if the above argument succeeds in showing that mathematics is not a creation of the human mind, it does not immediately follow that the objects of mathematics objectively exist. (I haven't decided yet whether I extensively examine the argument here. It would be more relevant to put the examination somewhere in the next chapter.)

So far, the target of Gödel's criticism has been referred as 'the view that mathematics is only our own creation."[52] It is right after the above argument that Gödel explicitly names the target of his criticism: conventionalism. He states the definition and implication of conventionalism as follows.

[50] [Gödel 1951, p. 314].

[51] [Gödel 1995, p. 298].

[52] [Gödel 1951, p. 311].

> It is that which interprets mathematical propositions as expressing solely certain aspects of syntactical (or linguistic) conventions, that is, they simply repeat parts of these conventions. According to this view, mathematical propositions, duly analyzed, must turn out to be as void of content as, for example, the statement "All stallions are horses." . . . Therefore the simplest version of the view in question would consist in the assertion that mathematical propositions are true solely owing to the definitions of the terms occurring in them, that is, that by successively replacing all terms by their definientia, any theorem can be reduced to an explicit tautology, $a = a$.[53]

Gödel objects to the view by saying that it is impossible to reduce all of mathematical propositions to tautologies because his incompleteness theorem prevents one from accomplish this reduction even as to arithmetic. However, besides its simplest version discussed above, there are more sophisticated version of conventionalism which could avoid Gödel's criticism. Thus, Gödel devoted his next paper to criticize conventionalism comprehensively.

5 "Is mathematics syntax of language?" (1953/1959)

In 1953, Gödel was invited to contribute a paper for the Rudolf Carnap volume of *The Library of Living Philosophers.* Paul Arthur Schilpp, who, as we have seen in the section two, was the editor of the series and had asked Gödel to write a paper for the Russell volume, proposed

[53] [Gödel 1951, pp. 315-316].

"Carnap and the ontology of mathematics" as his theme.[54] Although Gödel accepted this invitation and kept working on the paper for six years, he decided not to publish his paper after all because of difficulties in writing the paper. Gödel explains in his 1959 letter sent to Schilpp.

> The fact is that I have completed several different versions, but none of them satisfies me. It is easy to allege very weighty and striking arguments in favor of my views, but a complete elucidation of the situation turned out to be more difficult than I had anticipated, doubtless in consequence of the fact that the subject matter is closely related to, and in part identical with, one of the basic problems of philosophy, namely the question of the objective reality of concepts and their relations. On the other hand, in view of widely held prejudices, it may do more harm than good to publish half done work.[55]

Despite the fact that Gödel was not satisfied with any version of the paper and thought that "it may do more harm than good to publish half done work," the paper provides us with a more complete criticism of conventionalism than the one expressed in [Gödel 1951] and the clue to understanding Gödelian Platonism. In this section, we will examine how this criticism of conventionalism and try to figure out how it and its unsatisfactory result of the criticism are "closely related to . . . the question of the objective reality of concepts."

[54][Gödel 2003, p. 238].

[55][Gödel 2003, p. 244].

Gödel criticizes conventionalism primarily because it holds that "[m]athematics can be interpreted to be syntax of language"[56] and that "[m]athematical sentences have no content."[57] The second point can be thought as the premise of the first point. That is, conventionalism asserts that, if mathematics is about nothing and we still want to talk about the truth and falsity of mathematics, then we have to think mathematics as syntax of language.[58] In the unpublished manuscripts, Gödel first examines the first point and shows that what conventionalism asserts is wrong, and then proceeds to the second point. Because we have already seen in the previous section why Gödel thinks that the assertion that mathematics is syntax of language is wrong, we will concentrate on the second point here.

In criticizing the conventionalist view that mathematics has no content, Gödel begins his argument by dividing the view into two constituent parts, that is, 1) mathematical sentences have no empirical content and 2) "content" exclusively means "empirical content."[59]

---

[56] [Gödel 1953/9-III, p. 337].

[57] [Gödel 1953/9-III, p. 337].

[58] Seen in this way, conventionalism might seem to similar to formalism which is frequently thought of as "a mere game of symbols." However, as we noted in the footnote 5, formalism, especially that of Hilbert, does not regard mathematics as meaningless. Moreover, while conventionalists tends to think that mathematics can be totally reducible to syntax of language, that is, to logic, Hilbert does not think so. Hilbert says that "[n]o more than any other science can mathematics be founded by logic alone; rather, as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all though" ([Hilbert 1928, p. 44]).

[59] [Gödel 1953/9-III, p. 351].

Gödel completely agrees with the first of them.[60] It is the second point to which Gödel vehemently objects.

Gödel argues that the impossibility of reducing mathematics to syntax of language which Gödel has shown in the first part of the paper implies the falsity of the view that mathematical sentences have no content at all because

> if the prima facie content of mathematics were only a wrong appearance, it would have to be possible to build up mathematics satisfactorily without making use of this "pseudo" content.[61]

However, such an enterprise to build up mathematics in the conventionalist way now turns out to be unrealizable. In mathematical contents, there surely are parts which cannot be reduced to syntactical counterparts void of content.

Gödel also mentions another argument for the view that mathematics has no content. According to such an argument, "mathematics either is wrong or has no content, because if correct it is compatible with all possible sense experiences."[62] The key to fully appreciate this argument would be the quantified modal modifier *all possible*. Now suppose that the *actual* sense experience is X's whiteness. However, it is possible for X not to have had the property "whiteness." The sense experience could be X's non-whiteness. Thus, the set of the possible sense experiences as to X contains experiences of whiteness and non-whiteness.

---

[60][Gödel 1953/9-III, p. 351]. Gödel has already expressed the agreement to conventionalism in this point in [Gödel 1951]. See [Gödel 1951, p. 320].

[61][Gödel 1953/9-III, p. 346].

[62][Gödel 1953/9-III, p. 348].

Needless to say, if seen from the mathematical point of view, this set is inconsistent because it contains a property P and its negation at the same time. Therefore, for mathematics to be compatible with all possible sense experiences, it has to either contain inconsistency — in this case, mathematics is wrong — or have no overlap with all possible sense experiences — in this case, mathematics has no content.

Gödel objects to this argument by saying that

> that this inference is not valid even from the empirical standpoint follows from the fact that laws of nature without mathematics are exactly as "void" of content . . . as mathematics without laws of natures. The fact is that only laws of nature *together* with mathematics (or logic) have consequences verifiable by sense experience. It is, therefore, arbitrary to place all content in the laws of nature.[63]

We can find here the echo of what Gödel said in "Russell's mathematical logic." There, Gödel tried to show that even physical sciences cannot do without non-empirical elements. To construct a theory about or based on sense experiences, that is, to put sense data in order, we have to introduce what is not contained in or deducible from those data. As in this case, laws of nature must introduce some elements which cannot be found in nature. It is mathematics that provides such elements to laws of nature. Gödel talks about the relation between mathematics and physical theories in the fifth manuscript of "Is mathematics syntax of language?"

> [F]or a certain kind of physical theory a new mathematical axiom (which

[63] [Gödel 1953/9-III, pp. 348–349].

> would solve problems of mathematical physics formerly undecidable) may lead to new empirically verifiable consequences exactly as a new law of nature.[64]

It is easy to see that for Gödel a mathematical axiom is essentially and primarily about concepts. In fact, right after the above quotation, Gödel starts talking about concepts in mathematics and two kinds of content which conventionalism confuses.

> Mathematical propositions, it is true, do not express physical properties of the structure concerned, but rather properties of the *concepts* in which we describe those structures. But this only shows that the properties of those concepts are something quite as objective and independent of our choice as physical properties of matter. . . . However, in spite of the objective character of conceptual truth, it is quite necessary to distinguish sharply these two kinds of content and facts as "factual" and "conceptual."[65]

As easily seen from the above quotation, "concepts" play an important role in Gödelian Platonism. Actually, Gödel thinks concepts as the source of truth in mathematics.[66] However, at the same time, they are the very source of the difficulty which Gödel mentioned in the letter to Schilpp. To assert, against the conventionalist criticism, that mathematics does

[64][Gödel 1953/9-V, p. 360].

[65][Gödel 1953/9-V, p. 360]; italics in the original.

[66]In the same manuscript we are now examining, Gödel says that "mathematical propositions . . . are true in virtue of concepts occurring in them" ([Gödel 1953/9-V, p. 357]; italics in the original). We will return to this quotation in the next chapter and examine what it means to be true in virtue of concepts.

have contents which are said to be conceptual, Gödel should have made clear what it means to have conceptual contents and shown its objectivity. In the next chapter, we will examine these issues.

6 "The modern development of the foundations of mathematics in the light of philosophy" (1961)

In 1961, Gödel joined the American Philosophical Society. As was the custom, Gödel was supposed to give a talk at a Society meeting. "The modern development of the foundations of mathematics in the light of philosophy" is a manuscript for that talk, though it seems never to have been given.

Gödel starts his (virtual) talk by dividing the "philosophical world-views" (*Weltanschauungen*) into two groups "according to the degree and the manner of their affinity to or, respectively, turning away from metaphysics (or religion)."[67] To one side, which Gödel calls "the left," belong skepticism, materialism, and positivism; to the other belong spiritualism, idealism, and theology, which he calls "the right." Based on this distinction, Gödel argues that the history (or the development, as Gödel says) of philosophy since the Renaissance can be seen as the transition from the right to the left. On the other hand, according to Gödel, mathematics seems to have resisted this current from the right to the left, at least until the late nineteenth century. However, incited by the discovery of antinomies in set theory, the tendency to the left in mathematics has been gaining power.

Naturally, Gödel does not think well of this tendency. His dislike for the leftward

[67] [Gödel 1961, p.375].

tendency in mathematics seems to be rooted in the belief that if one takes the leftward stance toward mathematics, he or she discards the law of excluded middle and consequently the expectation that every proposition is either true or false.[68] Nonetheless, Gödel does not exclusively take the rightward position. He thinks that "the truth lies in the middle or consists of a combination of the two conceptions."[69] and that Hilbert is among those who tried to find such a combination.

At first glance, Hilbert might be thought of as one of the most earnest practitioners in the "left-wing" because Gödel's characterization of the leftward tendency in mathematics (the view which regards mathematics "as a mere game with symbols according to certain rules"[70]) is nothing but that of Hilbert's formalism. However, Hilbert also has the rightward tendency that "every precisely formulated yes-no question in mathematics must have a clear-cut answer."[71] Gödel argues that Hilbert's failure in finding the middle position between the right and the left is due to his strong tendency to the left. Actually, regardless of whether Hilbert really tried to find such a position or not, it is well-known fact that Hilbert's program cannot be realized at least in its original form exactly because of Gödel's incompleteness result.[72] Then, Gödel maintains, to secure the certainty of mathematics, we need to appeal

[68]It is not exactly that simple whether the leftward view of mathematics really implies the impossibility (or the prohibition) of the law of excluded middle.

[69][Gödel 1961, p. 381].

[70][Gödel 1961, p. 379].

[71][Gödel 1961, p. 379]. This Hilbert's rightward tendency might be also endorsed by the following famous words of Hilbert: "Wir müssen wissen. Wir werden wissen" ("We must know. We will know").

[72]Hilbert's program is, roughly speaking, the enterprise of founding mathematics on a solid and secure

to the rightward approaches, that is, "cultivating (deepening) knowledge of the abstract concepts"[73] and points out that Husserl's phenomenology can be used for such a purpose.

Gödel argues that to cultivate or deepen our knowledge of abstract concepts such as mathematical ones is not perfectly done only by giving "explicit definitions for concepts and proofs for axioms"[74] because such procedures have no end and moreover new axiom(s) might be needed for solving a certain type of problems such as the continuum problem. To amply cultivate our knowledge of mathematical concepts, Gödel maintains, we need to clarify "meaning[s] that [do] not consist in giving definitions."[75] And Gödel thinks that the clarification of meanings can be (at least partially) accomplished with the help of phenomenology. According to Gödel, such a clarification proceeds as follows.

> Here clarification of meaning consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts, onto our power in carrying out our acts . . . .[76]

Unfortunately, Gödel's explanation of the phenomenological method in clarifying meanings remains very sketchy. However, in connection with this phenomenological method

---

ground in the formalist way, that is, with a set of axioms and rules of inference. Gödel's incompleteness theorem shows that no matter what axioms and rules of inference we choose, there are mathematical statements which are true but cannot be proven with these axioms and rules of inference.

[73][Gödel 1961, p. 383].

[74][Gödel 1961, p. 383].

[75][Gödel 1961, p. 383].

[76][Gödel 1961, p. 383].

to secure the certainty of mathematics, it is worth mentioning the following comment of Gödel.

> [T]he whole phenomenological method, as I sketched it above, goes back in its ⟦central⟧ idea to Kant, and what Husserl did was merely that he first formulated it more precisely, made it fully conscious and actually carried it out for particular domain.[77]

We will extensively examine this issue, that is, the relation among the thoughts of Gödel, Husserl, and Kant on the epistemological issues of concepts in the third chapter. In the next chapter, we will turn to an investigation of concepts themselves, that is, of what Gödel thinks of them and what roles they play in Gödelian Platonism.

---

[77] [Gödel 1961, p. 383]; Double square brackets (⟦...⟧) are used for indicating annotations by editors.

Chapter 2
Concepts in Gödelian Platonism

As we glimpsed in the previous chapter, concepts play an extremely important role in Gödelian Platonism. However, it seems that this aspect of Gödelian Platonism had been neglected, or not received attention it deserves, at least until his *Collected Works* were published.[1] This neglect might have been partly because the paper where Gödel mainly talked about concepts was about Russell who once regarded concepts as real[2] and therefore this conceptual aspect of realism was thought of as Russell's. However, now we know from his formerly unpublished writings that Gödel kept talking about concepts until later in his career. The conceptual aspect is surely one of the main elements in Gödelian Platonism, and in fact he talked about concepts as an aspect of his own Platonism in [Gödel 1944].

In this section, we will try to make clear what Gödel thinks of concepts. Before dealing with these questions, let us first examine some preliminary points.

---

[1]One of the notable exceptions to this neglect is Paul Bernays' review of [Gödel 1944] ([Bernays 1946]), which we will examine later in this chapter.

[2]Actually, Russell says in his *Principles of Mathematics* that "*Being* is that which belongs to every conceivable term, to every possible object of thought—in short to everything that can possibly occur in any proposition, true or false, and to all such propositions themselves" ([Russell 1903, §427]; italics in the original). However, right after this characterization of "being," he also introduces another ontological distinction: existence. What Russell says about existence is just that it is necessary for something/someone to have some relation to this property called *existence* in order to exist ([Russell 1903, §427]). It seems from what he says in [Russell 1903, §434] that he means by "existence" the property of being in a given space at a given time, that is, being empirical in some sense.

1 Gödel and Quine

In the previous chapter, similarities between the thought of Gödel and Quine can be recognized. There are at least three similarities between them: the analogy between mathematics and physical sciences, the indispensability argument, and the holistic conception of theories. However, in fact, there is a fundamental and ineffaceable difference between the thought of Gödel and Quine. In this section, through the comparison between them, we will bring out one of the characteristics in Gödelian Platonism.

In his "On what there is," Quine says about the existence of external objects as follows.

> We should still find, no doubt, that a physicalistic conception scheme, purporting to talk about external objects, offers great advantages in simplifying our over-all reports. By bringing together scattered sense events and treating them as perceptions of one object, we reduce the complexity of our stream of experience to a manageable conceptual simplicity. The rule of simplicity is indeed our guiding maxim in assigning sense data to objects. . . .[3]

This is exactly what Gödel says in "Russell's mathematical logic." There, Gödel, by arguing about the role of positing the existence of physical objects in forming physical theories, asserts that the same can be said of mathematical objects in forming mathematical theories.[4] Quine

---

[3] [Quine 1948, p. 17].

[4] We have already cite the relevant part of [Gödel 1944] in the section two of the previous chapter. See p.

also follows a line of thought similar to Gödel's. He says:

> Physical objects are postulated entities which round out and simplify our account of the flux of experience, just as the introduction of irrational numbers simplifies laws of arithmetic. From the point of view of the conceptual scheme of the elementary arithmetic of rational numbers alone, the broader arithmetic of rational and irrational numbers would have the status of a convenient myth, simpler than the literal truth (namely, the arithmetic of rationals) and yet containing that literal truth as a scattered part.[5]

The wording "myth" in the above quotation might make one suspect that Quine does not really believe in the existence of mathematical objects. However, the use of "myth" here should be regarded just as *façon de parler*. In fact, Quine uses the same word about physical objects.[6] He just pretends to be a phenomenalist who regards the existence of some objects, whether those objects are physical or mathematical, as a myth or illusion. The whole point is that, even for phenomenalists, if one admits a certain property about physical objects and if relevant conditions are the same for mathematical objects,[7] he or she must admit the same property about mathematical objects. Thus, if physical objects can be postulated for

---

11 of this thesis.

[5][Quine 1948, p. 18].

[6][Quine 1948, p. 18]. There, he says that "the conceptual scheme of physical objects is a convenient myth."

[7]In this case, the condition needed to be the same would be to postulate some entities for simplifying a theory.

simplifying physical theories, mathematical objects can be also postulated for simplifying mathematics.

This analogy concerning the positing of some objects for simplifying (or advancing) theories, can be pushed further. That is, not only physical objects but mathematics itself are needed for forming simplified and effective physical theories. In other words, physical theories presuppose the existence of mathematics. Then, if admitting physical theories, one must also admit the mathematics. This is the so-called "indispensability argument" which stemmed from arguments of Quine and was developed by Putnam. As to the indispensable role of mathematics for physical sciences, Quine says:

> A platonistic ontology of this sort [i.e., classes or attributes of physical objects] is, from the point of view of a strict physicalistic conceptual scheme, as much a myth as that physicalistic conceptual scheme itself is for phenomenalism. This higher myth is a good and useful one, in turn, in so far as it simplifies our account of physics. Since mathematics is an integral part of this higher myth, the utility of this myth for physical science is evident enough.[8]

Here Quine argues that mathematics is embedded in physical sciences in an indispensable

[8] [Quine 1948, p. 18].

way.[9] This is what Gödel also maintains in [Gödel 1953/59-III].[10] Removing mathematics from physical sciences amounts to the loss of a considerable part of physical sciences, if not all of them.[11]

The indispensability argument has, especially for Quine, another implication. More properly, the argument is based on another view: the holistic conception of theories. And this view leads to Quine's famous rejection of the analytic/synthetic distinction. This rejection is what parts Gödel from Quine. To see the difference between Gödel and Quine clearly, let us take a brief look at how Quine advances his argument for the rejection of the analytic/synthetic distinction.

Although Quine tries to defend his view in several ways, we concentrate on one of his

---

[9]Quine himself does not use the word "indispensable" to express this relation between physical sciences and mathematics. It is Putnam who clearly uses the word for that purpose. Putnam says that "quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities" ([Putnam 1971, p. 56]).

[10]We have already cite the relevant part of [Gödel 1953/59-III] in the section 5 of the previous chapter. See p. 29 of this thesis.

[11]Needless to say, there are objections to the indispensability argument. Although we cannot fully examine these objections here, it is fair to mention one of these. Among the objections to the indispensability argument, Hartry Field's seems most thorough. In his *Science without Numbers*, Field actually reconstructs Newtonian mechanics without using mathematics. However, Field's work just shows the dispensability of mathematics only as to Newtonian mechanics. As to other parts of physical sciences, mathematics might still be indispensable (for example, quantum mechanics seems to definitely need mathematics in its theoretical construction). For more about the indispensability arguments, see [Colyvan 2001].

strategies. Quine maintains that reductionism supports the analytic/synthetic distinction.[12] Therefore, if reductionism is refuted, so is the distinction. And it is the holistic conception of theories that Quine thinks of as what refutes reductionism.

According to Quine, reductionism is the view that "[e]very meaningful statement is held to be translatable into a statement (true or false) about immediate experience."[13] Quine regards Carnap, especially his *Der logische Aufbau der Welt*, as representative of this view and thinks that the view cannot not be held anymore because the program of the *Aufbau* turned out to be a failure. However, Quine argues that reductionism (or the dogma of reductionism) has survived in a different guise, that is, "in the supposition that each statement, take in isolation from its fellows, can admit of confirmation or infirmation at all."[14] Holism about theories attacks the very supposition.

The holistic view about theories is well described in Pierre Duhem's *La théorie physique* as Quine points out.[15] Duhem summarizes why the above supposition cannot be held as follows.

> In sum, a physicist cannot test a hypothesis in isolation but a whole set of

[12] ". . . the one dogma [reductionism] clearly supports the other [the analytic/synthetic distinction] in this way: as long as it is taken to be significant in general to speak of the confirmation and infirmation of a statement, it seems significant to speak also of a limiting kind of statement which is vacuously confirmed, *ipso facto*, come what may; and such a statement is analytic" ([Quine 1951, p. 41]; italics is in the original). We should note that Quine identifies "analytic" with "a priori" in this quotation.

[13] [Quine 1951, p. 38].

[14] [Quine 1951, p. 41].

[15] See the footnote 15 in [Quine 1951, p. 41].

> hypotheses because if an experiment is not accord with its prediction, it tells the physicist that at least one of its hypotheses is unacceptable and to be modified. However, the experiment does not tell which hypothesis is to be changed.[16]

Note that hypotheses in an experiment do not need to be limited to empirical hypotheses, although what Duhem himself has in mind in this context is mainly empirical ones. In fact, Quine extends what Duhem says so that hypotheses can also contain theoretical ones such as mathematical statements. Thus, to admit the holistic conception of theories suggests that one must discard the distinction between empirical and mathematical statements. However, for Gödel, this cannot be admitted at all. Gödel says:

> The syntactical point of view as to the nature of mathematics doubtless has the merit of having pointed out the fundamental difference between mathematical and empirical truth. This difference, I think rightly, is placed in the fact that mathematical propositions, as opposed to empirical ones, are true in virtue of the *concepts* occuring in them.[17]

For Gödel, the difference between mathematical and empirical truth is fundamental. However, it seems that to be true in virtue of concepts means to be analytically true. Then, how is it possible for Gödel to keep the concept of "analyticity" and the indispensability ar-

[16] [Duhem 1914, p. 284]; my translation. Despite what Quine says, Carnap willingly admits this Duhem's view. He states in his *Logical Syntax of Language* that "the test applies, at bottom, not to a single hypothesis but to the whole system of physics as a system of hypotheses (Duhem, Poincaré)" ([Carnap 1937, p. 318]).

[17] [Gödel 1953/59-V, pp. 356-357].

gument at the same time if the argument is based on the rejection of the analytic/synthetic distinction? Moreover, how is it possible for mathematical statements to be analytically true and have meaningful contents at the same time? The key to answer these questions is in Gödel's very conception of analyticity. In the next section, we will examine the concept of analyticity according to Gödel.

2 Gödel's conception of analyticity

In the fifth manuscript of "Is mathematics syntax of language?" Gödel says the following concerning the distinction between mathematical and empirical truth.

> The syntactical point of view as to the nature of mathematics doubtless has the merit of having pointed out the fundamental difference between mathematical and empirical truth. This difference, I think rightly, is placed in the fact that mathematical propositions, as opposed to empirical ones, are true in virtue of the *concepts* occurring in them.[18]

Reading the above quotation, one might ask the following series of questions: If mathematical propositions are true in virtue of concepts, does that mean that they are true in virtue of meanings of words which occur in the sentences expressing them? Then, does it imply that mathematical truth is analytical? If this chain of inference is right, it seems to follow that mathematical propositions, if true, do not have any content, just as conventionalism asserts. How is it possible that, as Gödel maintains, mathematical propositions are

[18] [Gödel 1953/1959-V, pp. 356-357]; italics in the original.

analytical and have contents at the same time?

As to the first point, that is, that mathematical propositions are true in virtue of concepts means that they are true in virtue of meanings of words in propositions, recall what Gödel said in [Gödel 1961]. There, Gödel virtually identified "concepts" with "meanings."[19] True, the "meaning" argued in [Gödel 1961] is supposed not to "consist in giving definitions."[20] This might seem to imply that the above identification of concepts with meanings of words in mathematical propositions is wrong. However, Gödel explains what he means by "concept" in [Gödel 1944] as follows.

> Classes and concepts may, however, also be conceived as real objects, namely classes as "pluralities of things" or as structure consisting of plurality of things and concepts as the properties and relations of things *existing independently of our definitions* and constructions.[21]

Taking the above quotation into account, some words in mathematical propositions have meanings which are the properties and relations of mathematical objects existing independently of us. This interpretation coexists with what is said in [Gödel 1961]. However, some might object that it is just an anachronism to interpret an text with the help of another text which is written more than ten years before. In fact, Gödel's conception of "concept" seems to have had been unchanged throughout his life. Let us compare the above quotation with the following.

---

[19] [Gödel 1961, p. 383]. See the section 6 in the previous chapter.

[20] [Gödel 1961, p. 383].

[21] [Gödel 1944, p. 128]; italics added.

> Mathematical propositions . . . do not express physical properties of the structures concerned, but rather properties of the *concepts* in which we describe those structures.[22]

> [Axioms of set theory] cannot be reduced to anything substantially simpler, let alone to explicit tautologies. It is true that these axioms are valid owing to the meaning of the term "set" — one might even say they express the very meaning of the term "set" . . . .[23]

Given these quotations, we can think that mathematical propositions' being true in virtue of concepts can imply (or be indentified with) their being true in virtue of meanings of words in the sentences expressing them.

Next, how about the second point, that is, that propositions are true in virtue of meanings implies their analyticity? On the very surface, this implication seems undoubtedly true. Then, mathematical propositions must have no content. However, for Gödel, meanings, especially those of words in mathematics, are not what we freely posit but what exist independently of us. From this conception of meanings, Gödel's seemingly peculiar conception of analyticity follows.

In the conclusion of [Gödel 1944], Gödel argues the analyticity of the axioms of *Principia Mathematica*. It has long been disputed whether the axioms of *Principia*, especially the axiom of reducibility, are logically true, that is, necessarily true.

---

[22] [Gödel 1953/59-V, p. 360]; italics in the original.

[23] [Gödel 1951, p. 321].

The axiom of reducibility is, roughly speaking, that which asserts that all mathematical propositions can be transformed into those which do not contain impredicative definitions even if we cannot actually find such transformed propositions.[24] It is easy to see that the axiom is non-constructive as the axiom of choice is.[25] Therefore, it is no wonder that many people has questioned its status as being logically true. Actually, as early as 1922, Ludwig Wittgenstein criticizes this axiom as follow.

> The general validity of logic might be called essential, in contrast with the accidental general validity of such propositions as 'All men are mortal'. Propositions like Russell's 'axiom of reducibility' are not logical propositions, and this explains our feeling that, even if they were true, their truth could only be the result of a fortunate accident.[26]

> It is possible to imagine a world in which the axiom of reducibility is not valid. It is clear, however, that logic has nothing to do with the question whether our world really is like that or not.[27]

Following this criticism of Wittgenstein, Frank P. Ramsey also criticizes the axiom.

> This axiom there is no reason to suppose true, and if it were true, this would be a happy accident and not a logical necessity, for it is not a tautology.

---

[24] As to impredicative definitions, see the section 1 of the previous chapter.

[25] As to the problem of the axiom of choice, see the section 1 of the previous chapter.

[26] [Wittgenstein 1922, 6.1232].

[27] [Wittgenstein 1922, 6.1233].

> . . . Such an axiom has no place in mathematics, and anything which cannot be proved without using it cannot be regarded as proved at all.[28]

If these critiques of the axiom of reducibility are right, *Principia*, which contains the axiom as its indispensable part, loses its status as the program of logicism. Gödel, by distinguishing two meanings of "analyticity," tries to save *Principia* from these criticisms.

> As to this problem [i.e., the problem whether the axioms of *Principia* are analytical or not], it is to be remarked that analyticity may be understood in two senses. First, it may have the purely formal sense that the terms occurring can be defined (either explicitly or by rules for eliminating them from sentences containing them) in such a way that the axioms and theorems become special cases of the law of identity and disprovable propositions become negations of this law.[29]

In this sense, Gödel argues, even the theory of integers is not analytical. If the theory of integers is analytical, there must exist an elimination procedure of finite length for each proposition and axiom in the theory. The existence of such a procedure for each proposition and axiom in the theory implies that of a decision procedure for all propositions and axioms in the theory. This is, however, what Gödel's theorem proved impossible.[30] Therefore, the

---

[28] [Ramsey 1926, pp. 358-359]

[29] [Gödel 1944, pp. 138-139].

[30] Gödel himself refers to [Turing 1937] as what shows the above implication ([Gödel 1944, p. 139]). Needless to say, it can be shown that the result of Turing and Gödel's theorem are equivalent in a strict sense. As to the equivalence of the result of Church and Gödel's theorem, see the section 60 of [Kleene 1952].

axioms of *Principia* which are supposed to be sufficient to capture the whole of classical mathematics are not analytical in this sense of "analyticity."

In the above explanation, it is said that for a proposition (or an axiom) to be analytical, there must exist an elimination procedure of a finite length for the proposition. The phrase "finite length" is crucial here. It is true that if one admits an elimination procedure of *infinite* length, he or she can show the analyticity of all axioms of *Principia*.[31] However, Gödel points out that "the whole of mathematics . . . has to be presupposed in order to prove this analyticity."[32] In other words, each axiom in the system should be regarded as true before one shows its analyticity. This is, of course, far from a satisfactory solution. Then, in the end, can the axioms of *Principia* not be regarded as analytic? Gödel asserts that according to the second sense of "analyticity," they are analytical.

> In a second sense a proposition is called analytic if it holds "owing to the meaning of the concepts occurring in it," where this meaning may perhaps be undefinable (i.e., irreducible to anything more fundamental). It would seem that all axioms of *Principia*, in the first edition, (except the axiom of infinity) are in this sense analytic . . . .[33]

This conception of "analyticity" that distinguishes "true by meanings of concepts" from "true by definition" is certainly not standard. It might be regarded even as ad hoc.

[31] This is the direction Ramsey took in [Ramsey 1926]. However, as Gödel points out in a footnote ([Gödel 1944, p. 139]), Ramsey failed to show the analyticity of the axiom of infinity.

[32] [Gödel 1944, p. 139].

[33] [Gödel 1944, p. 139].

However, by conceiving "analyticity" in this way, Gödel can maintain that mathematics is analytical and has contents at the same time.[34] And he retains this conception until later. Actually, in 1951, he writes as follows.

> I wish to repeat that "analytic" does not mean "true owing to our definitions," but rather "true owing to the nature of the concepts occurring ⟦therein⟧," in contradistinction to "true owing to the properties and the behaviour of things." This concept of analytic is so far from "void of content" that it is perfectly possible that an analytic proposition might be undecidable (or decidable only with ⟦a certain⟧ probability).[35]

[34]And of course, from this conception of "analyticity," it follows that the asserting indispensability of mathematics for physical sciences does not necessarily means rejecting the analytic/synthetic distinction because "analyticity" in the analytic/synthetic distinction which is to be rejected in the (usual) indispensability argument is not the same as in the Gödelian sense. Moreover, Gödel seems to think of the different basis for his indispensability argument. We will come back to this point in the next chapter.

[35][Gödel 1951, p. 321]. Following this quotation, Gödel explains the reason why some analytical propositions might be "decidable only with ⟦a certain⟧ probability" as follows. "For, our knowledge of the world of concepts may be as limited and incomplete as that of ⟦the⟧ world of things" ([Gödel 1951, p. 321]). This view about decidability conforms with the view about undecidable propositions expressed in [Gödel 1947]. In [Gödel 1947], Gödel says about undecidable propositions that the fact that we do not have a formal proof of some proposition *now* does not deny at all that we might have such a proof *some day in the future*. Moreover, even when we do not have a formal proof of a proposition, we sometimes have a kind of *likelihood* about whether such a formally undecidable proposition is true or not. For example, although the continuum hypothesis (the conjecture that $2^{\aleph_0} = \aleph_1$) is still undecidable with the current set of axioms of set theory, we can have an impression that the continuum hypothesis is actually wrong, that is, an impression that $2^{\aleph_0} \neq \aleph_1$ from results such as [Woodin 2001]. In short, the more results we get about an currently

With this conception of "analyticity," there is nothing peculiar in regarding mathematical propositions and axioms as analytical and as having contents. However, his very conception of "analyticity" is nothing but peculiar. Does this conception really capture the characteristics of what is called "analyticity"? To answer this question properly, we have to dig deeper into what the conception of "concept" Gödel has in mind.

3 Gödel's conception of concepts

What is a concept? This is surely a tough question to answer. This toughness seems to be because of the essential vagueness in the word. Due to this vagueness, each uses the word in his or her own way. Then, how does Gödel use the word? To answer this question, let us begin with the deliberately untouched aspect of Gödel's thought in [Gödel 1944]: Gödel's conception of "concepts."

As we have already seen in the previous section, Gödel thinks that concepts are "conceived as real objects" as well as classes are and "as the properties and relations of things existing independently of our definitions and constructions."[36] We have also seen that it is thanks to the meanings of such concepts that a proposition can be regarded as analytical even when the proposition cannot be regarded as analytical in an ordinary sense, that is, true only by the meanings of words appearing in it. In conceiving analyticity so, mathematical propositions can be regarded as analytical and meaningful at the same time. Following this characterization of analyticity, Gödel says more about concepts as follows.

---

undecidable proposition, the deeper our conviction about whether the proposition is true or not becomes.

[36] [Gödel 1944, p. 128].

> It would seem that all the axioms of *Principia*, in the first edition, (except the axiom of infinity) are in this sense [i.e., in the conception of analyticity as explained above] analytic for certain interpretations of the primitive terms, namely if the term "predicative function" is replaced either by "class" (in the extensional sense) or (leaving out the axiom of choice) by "concept," since nothing can better express the meaning of the term "class" than the axiom of classes . . . and the axiom of choice, and since, on the other hand, the meaning of the term "concept" seems to imply that every propositional function defines a concept.[37]

To fully appreciate what Gödel says in the above quotation, we need to explain what "predicative function" and "propositional function" mean.

In the Russellian terminology, or in *Principia*, a propositional function means

> something which contains a variable x, and expresses a proposition as soon as a value is assigned to x. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned.[38]

For example, let us think of the propositional function "A is A." By assigning a value, say, Socrates, to A, we get the sentence "Socrates is Socrates," which is, needless to say, a tautology. The meaning of the resulting sentence seems quite obvious: That Socrates

[37] [Gödel 1944, p. 139].

[38] [Russell and Whitehead 1927, p. 38].

is Socrates.[39] Then, what is the meaning of the propositional function "A is A" itself? It cannot be said that the meaning of the propositional function "A is A" is that A is A because, unlike "Socrates," the variable A does not mean anything specific (or means almost everything) and consequently the propositional function "A is A" cannot have a definite meaning. However, something makes us hesitate to say that a propositional function does not have any meaning at all. That is why Russell says that "it is ambiguous." Gödel goes beyond saying that it is ambiguous. He says

> [A propositional function is] something separable from the argument (the idea being that propositional functions are abstracted from propositions which are primarily given) and also something distinct from the combination of symbols expressing the propositional function; it is then what one may call the notion or concept defined by it.[40]

In this conception, the above propositional function "A is A" is interpreted as expressing the concept of self-identity. Thus, according to Gödel, it is said that "every propositional function defines a concept."

The explanation of "predicative function" needs the concept of "order." A function whose arguments are all individuals is called *first-order*.[41] A second-order function is a

---

[39]In fact, the meaning of a sentence is not so obvious. For example, from the Fregean point of view, the meaning of a sentence is divided into two: sense and reference. The sense of a sentence is the same as the meaning of a sentence explained in the body. In other sense, the sense of a sentence is the truth condition of the sentence. The reference of a sentence is simply a sentence's truth value.

[40][Gödel 1944, p. 124].

[41]An individual can be thought of as a "zeroth-order" function which receives no argument and outputs

function whose arguments are either first-order functions or individuals. In general, an nth-order function is a function the highest order of the arguments of which does not exceed n. A predicative function is a function the highest order of the arguments of which does not exceed the order of the function. Needless to say, something impredicative discussed in the previous chapter cannot be an instance of predicative function.

With the above clarifications, let us examine the analyticity of the axiom of reducibility. Informally, as we saw in the previous chapter, the axiom of reducibility asserts that for every proposition there is its predicative version. More formally, the axiom can be written as follows.

$$\forall\phi\exists\psi\forall x(\phi x \equiv \psi!x)$$ [42]

In the above, ϕ stands for any propositional function and $\psi!$ stands for any predicative function. Now, from what is said above ("every propositional function defines a concept"), we can identify a propositional function with a certain concept which is supposed to be defined by the function. Moreover, we can also identify a predicative function with a certain concept. Taking these identifications into account, what the axiom of reducibility asserts is: for every concept, there is a concept which has the same content as the original one. For there is no limitation about the choice of concepts, we can express the reinterpreted version of the axiom in the following quasi-formal way.

$$\forall\phi\forall x(\phi x \equiv \phi x)$$

---

a certain value.

[42] [Russell and Whitehead 1925, p. 56]. The original notation is $\vdash : (\exists\psi) : \phi x \,.\equiv_x .\, \psi!x$.

In this case, ϕ stands for any concept. Then, the axiom is simply a tautology, that is, "any concept is identical to itself." Actually, in a footnote to where he talks about his view of analyticity, Gödel says:

> It is to be noted that this view about analyticity makes it again possible that every mathematical proposition could perhaps be reduced to a special case of $a = a$, namely if the reduction is effected not in virtue of the definition of the term occurring, but in virtue of their meaning, which can never be completely expressed in a set of formal rules.[43]

However, there seems a gap or a *petitio principii* in Gödel's argument.

Gödel's argument that all the axioms of *Principia* are analytical starts with the assumption that "every propositional function defines a concept." Then, based on this assumption, it is asserted that the term "predicative function" can be replaced by "concept" because the set of predicative function is a subset of the set of propositional function. However, to properly replace the term "predicative function" by "concept," this "concept" must be predicative. How is it assured?

The first possibility is to think that every propositional function is predicative. Accordingly, every concept defined by such a propositional function is predicative. There is nothing wrong with replacing the term "predicative function" by such a predicative concept. But this possibility seems untenable because Gödel *does* admit the existence of impredicative propositions.[44]

---

[43][Gödel 1944, p. 139].

[44]Moreover, if every propositional function were predicative, the axiom of reducibility would be perfectly

The second possibility is to presuppose the axiom of reducibility. If presupposing the axiom, there surely is the predicative version of a propositional function. If there is the predicative version of a propositional function, so is there for a concept defined by that propositional function. However, needless to say, this kind of *petitio principii* cannot be admitted at all.

The last possibility is that for Gödel, even though a propositional function which defines a concept is impredicative, the concept thus defined can be thought of as predicative, or more daringly, as existing in the realm where the distinction predicative/impredicative becomes meaningless. This possibility seems to conform with the following thoughts of Gödel about the relation of mathematical reality and its expressions.

First, Gödel thinks that mathematical reality can be captured only incompletely in a formal system[45] because mathematical reality allows impredicativity, while a formal system does not.[46] Therefore, at least in a practical sense, taking about the distinction predicative/impredicative has no meaning in mathematical reality because any definition, whether it is predicative or not, is allowed there.

Moreover, Gödel believes that mathematical propositions (including axioms) can be

---

no use, even though the existence of the axiom does not do any harm at all.

[45]This statement might remind one of his incompleteness theorem. This is partly right because Gödel actually thinks that there is a gap between mathematical reality and its formal expression. The statement of the first incompleteness theorem ("there exist mathematical propositions which are true, nut not provable") can be interpreted as backing up such a thought. As to this point, see the section 4 of the previous chapter.

[46]Recall that Gödel said that there is no problem in defining something in a impredicative way. Actually, Gödel thinks that impredicative definitions are necessary for mathematics. As to this point, see the section 2 of the previous chapter.

regarded as true by means of concepts which appear in them. In other words, mathematical propositions are true if and only if they properly capture concepts which, according to Gödel, can provide more accurate pictures of mathematical reality than a formal system can.[47] Thus, mathematicians should try hard to capture mathematical reality which is comprised of classes and concepts by means of a formal system in a paradox-free way. And Gödel believes that this is possible by clarifying concepts.[48]

So far, we have not paid an attention to the expression "the meaning of concept" which Gödel frequently uses. It might be suggested that terms such as "concept-word" or "concept-term" should be used instead of "concept" because the term "concept" itself seems to have the same meaning as "meaning." However, even if we interpret the term "concept" as "concept-word" or "concept-term," there still remains a problem as Bernays points out in his review of [Gödel 1944] — that is, the problem of how we should understand the meaning of concept-word.

After bringing up the Frege's distinction of sense (Sinn) and reference (Bedeutung), Bernays argues as follows.

> [S]ince signification [reference] concerns the confrontation of our notions and propositions with the world of facts, whereas sense has to do with the inner

[47] That mathematical proposition are true by means of concepts in them seems inconsistent with the claim that "every propositional function defines a concept." However, for Gödel, it is an undeniable truth that mathematical concepts precede mathematical propositions because mathematical concepts exist independently of us and therefore we cannot define such an existence in an ordinary sense of "define." We should interpret the term "define" as meaning "capture."

[48] We will examine this clarification of concepts in the next chapter.

> content of the notions and propositions, as intended (expressed) by terms and sentences, considerations for analyzing conception have to deal with sense, not with signification.[49]

Bernays criticizes Gödel for confusing two meanings of "meaning" and says that it is exactly where Gödel distinguishes two kinds of analyticity that he confuses two meanings of "meaning." Bernays thinks that Gödel uses "meaning" as "reference" in his second characterization of analyticity according which a proposition is analytical if it is true by means of "the meaning of the concepts occurring in it." However, according to Bernays, Gödel should use "meaning" as "sense" in his second characterization of analyticity. Bernays refers to the Gödel's claim that mathematical propositions can be reduced to the tautology $a = a$ if the proposition is considered in terms of concepts occurring in them as the example of Gödel's confusion.

> [T]ransformations by which arbitrary mathematical propositions can be reduced to special cases of $a = a$ surely will not preserve the same sense of the sentences in question; so the reduction "in virtue of the meaning" can only be in virtue of the extensional meaning, i.e., by steps having a like a character to that of replacing the proposition, "The author of Waverley is Walter Scott," by "Walter Scott is Walter Scott."[50]

---

[49] [Bernays 1946, p. 78]. Bernays also maintains, following this quotation, that by regarding "analyzing conception" as dealing with sense, "the difficulties and paradoxical statements in the discussion about descriptions . . . all disappear."

[50] [Bernays 1946, p. 78].

Does Bernays' criticism do justice to Gödel's argument that "arbitrary mathematical propositions can be reduced to special cases of $a = a$"? In the following, paying attention to Gödel's conception of "concept" and comparing his conception of concepts in 1944 to his later conception, let us examine this issue.

First, note that Bernays seems to take the reduction in a formal sense. In other words, for Bernays, the reduction is done by replacing a term with another term which has the same reference as the former. However, what Gödel has in mind as to the reduction seems to be something different. Gödel clearly states that the reduction cannot be accomplished completely in a formal way. This means that the reduction Gödel speaks of is not what Bernays takes as the reduction. Then, what kind of reduction has Gödel in mind? Unfortunately, Gödel does not provide many details about this reduction is in [Gödel 1944]. However, we can guess what kind of reduction Gödel has in mind from what he says in [Gödel 1944] or other papers about concepts.

Recall that Gödel thinks that a propositional function can be identified with a certain concept in some sense. On the other hand, he also thinks that a concept cannot be completely expressed in a formal way. This means, despite Gödel's saying that "every propositional function defines a concept," a concept can be only partially expressed by a certain propositional function.[51] Moreover, as implied by what Gödel says in [Gödel 1947], a concept seems to have a close relation to other concepts.[52] Then, the reduction of a mathematical

---

[51] We already mentioned this point in the above. Taking this relation between propositional function and concept into account, Gödel should have said that every propositional function present a partial picture of a concept.

[52] "[T]he concepts and axioms of classical set theory . . . describe some well-determined reality" [Gödel

proposition to a special case of $a = a$ cannot be done simply by replacing a concept occurring in the proposition with another concept. To execute such a reduction, we have to take the whole set of concepts into account in a way that we will examine below.[53]

From the above considerations, we can gain some insights into what the term "concept" means for Gödel. First of all, Gödel seems to think that there is a gap between mathematical reality and its formal expressions and that the clarification of concepts plays an important role in filling such a gap.

According to Gödel, mathematical reality can be captured by formal devices only partially and incompletely. This conception of mathematics and its formalization is consistent with Gödel's later thought that we might be able to perceive and then formalize mathematical reality only incompletely. Actually, in his "Some basic theorems of the foundations of mathematics and their implications," Gödel says that "mathematics describes a non-sensual reality, which exists independently both of the acts and ⟦of⟧ the disposition of the human mind and is only perceived, and probably perceived very incompletely, by the human mind."[54] He also implies in the same paper that in order for mathematics to describe

---

1947, p. 181].

[53]In her [Crocco 2006], Gabriella Crocco argues that Bernays misunderstands what Gödel says in [Gödel 1944] by interpreting "meaning" within the Fregean framework. Within this framework, the meaning of a concept-word is divided into its sense, i.e., the concept itself and its reference, i.e., objects falling under the concept. However, according to Crocco, Gödel does not think of "meaning" within the Fregean framework. Crocco maintains that the reference of a concept-word is the concept itself and that this interpretation in fact accords with what Frege thinks about the meaning of a concept-word. Although we admit that Crocco's interpretation is plausible and insightful, we believe that ours is simpler and more straightforward.

[54][Gödel 1951, p. 323].

such a reality, it is necessary to clarify "the properties and relations of things" in that reality, that is, the concepts about reality.[55] This line of thought about the need for clarification of such concepts had remained central for solving mathematical problems later in Gödel's career. In fact, he argues about the necessity of the clarification of the concepts again in his [Gödel 1961].[56] This means, in turn, that Gödel still thought in 1961 that we can perceive and consequently formalize mathematical reality only incompletely.

Another insight we can draw from the counter-argument against Bernays' criticism is that Gödel thinks that a concept should be understood in relation to other concepts. It is almost meaningless to talk about a concept in isolation. Recall that Gödel defines "concepts" as "the properties and relations of things."[57] and also note that in mathematics what are called "things" are usually abstract objects like "sets" or "numbers." Although Gödel willingly admits the objective existence of such mathematical objects,[58] they are virtually identified with the set of concepts in most cases. Actually, in [Gödel 1951], Gödel says that the axioms of set theory express the very meaning of the term "set."[59] When we talk about "set," we actually talk about the collection of concepts which express the meaning

[55][Gödel 1951, p.322]. Actually, Gödel already said almost the same thing in [Gödel 1944]: "the primitive concepts need further elucidation" ([Gödel 1944, p. 140]).

[56]See the section 6 of the previous chapter.

[57][Gödel 1944, p. 128].

[58]"Classes and concepts may . . . also be conceived as real objects" ([Gödel 1944, p. 128]).

[59][Gödel 1951, p. 321]. Here we identified "concepts" with "axioms." This identification would be justified by seeing that the axioms of set theory can be regarded as propositional functions and that Gödel thinks that a propositional function "defines" a concept.

of "sets." Therefore, to think about the analyticity of each axiom in set theory, we have to have all axioms in mind. Axioms, or concepts, are thus interrelated with each other.

As we have seen above, for Gödel, concepts are first of all what can precisely capture mathematical reality. Thus, a mathematical proposition (expressed in some formal system) can be said to be analytical in the same sense as "water is H_2O." As in the case that water turned out to be H_2O, to advance our knowledge of mathematics, we need to first clarify what concepts are.[60] Moreover, in Gödel's understanding, concepts are interrelated with each other. This is so even if the interrelated concepts seem simple. For example, let us think of the concept of "set." As is well known, "set" is expressed by means of the collection of axioms each of which expresses a concept.[61] These axioms, interrelated with each other, express the class and concept of "set." With these characteristics of concepts in mind, we should clarify concepts in order to make (a formal expression) of mathematics more secure and more precise. Then, how should we do for clarifying concepts? To answer this question, we need to examine another important characteristic of Gödelian Platonism: intuition. This will be our objective in the next chapter.[62]

---

[60] It might be relevant to recall here that Gödel admits inductive or experimental method in mathematics. This seems due to this conception of concepts as objective existence in mathematical reality.

[61] Moreover, an axiom which expresses a concept can perhaps contain other concepts. This multiplies the complexity of interrelatedness of concepts.

[62] Before moving to Chapter 3, we should briefly mention Gödel's view about the ontological status of concepts. Although it is certain that Gödel believes in the objective existence of concepts as well as in that of classes, Gödel has never dogmatically asserted the objective existence of concepts. He always asserted it in a conditional form that if we admit the objective existence of physical object, then we should also admit that of abstract objects such as mathematical ones.

Chapter 3
Mathematical Intuition

“Intuition” is one of the most important and, at the same time, most controversial features in Gödelian Platonism. It is important because it allegedly provides a link which connects us mere humans to mathematical objects. And it is controversial because for some interpreters this feature represents a “mystical” aspect in Gödelian Platonism. According to such interpreters, Gödelian intuition is regarded as a mysterious faculty which enables us to directly access mathematical objects. For example, Chihara, based on this conception of intuition, makes a harsh assessment of Gödelian Platonism and suggests taking other approaches than Gödel’s to the problem of mathematical existence.

> Gödel’s appeal to mathematical perceptions to justify his belief in sets is strikingly similar to the appeal to mystical experience that some philosophers have made to justify their belief in God. . . . It is not surprising that other approaches to the problem of existence in mathematics have been tried.[1]

Even those more sympathetic to Gödel’s view think of Gödelian intuition as somewhat implausible, if not mystical or mysterious. For example, in her book which shows sympathy for the realistic conception of mathematics, Maddy writes about Gödelian intuition as follows.

> [A] faculty of mathematical intuition . . . plays a role in mathematics analogous to that of sense perception in the physical sciences, so presumably the axioms force themselves upon us as explanations of the intuitive data

[1] [Chihara 1990, p. 21].

> much as the assumption of medium-size physical objects forces itself upon us as an explanation of our sensory experiences.[2]

She identifies the objective of a chapter of her book as replacing Gödelian intuition with her naturalistic epistemology and naturalizing Gödelian Platonism.[3]

Despite their overall different attitudes toward Gödelian Platonism — Maddy is sympathetic to Gödelian Platonism and Chihara is not — those two interpreters share a view on Gödelian intuition: it is a faculty through which we can supposedly access mathematical objects and without the presupposition of which a theory of mathematical existence should be constructed. Moreover, they both refer to the same paragraph from [Gödel 1944] as the evidence of their judgment.[4] As we have already examined in the section 2 of the previous chapter, however, the paragraph which Chihara and Maddy regard as assuring their argument cannot be taken as such. Actually, in that paragraph, Gödel does not talk about mathematical intuition at all. The main point which Gödel argues there is that we have to presuppose the existence of mathematical objects in some sense in order to formulate a theory about mathematical "data." It is not about how such data can be known, that is, about mathematical intuition itself.

Moreover, there is another problem in interpreting Gödelian intuition: identifying the object of Gödelian intuition almost exclusively with mathematical objects such as sets and numbers, that is, with classes as "pluralities of things." Such an interpretation is evident

---

[2] [Maddy 1990, p. 31].

[3] [Maddy 1990, p. 35].

[4] See [Chihara 1990, p. 17] and [Maddy 1990, p. 32]. The paragraph mentioned here is cited on page 9 of this paper.

in Chihara's remarks. For example, in the section where he examines Gödelian platonism, Chihara almost entirely uses the phrase "mathematical objects as sets" for expressing the objects of mathematical intuition.[5] However, especially in the context Chihara has in mind, it is not the case that the objects of Gödelian intuition are exclusively classes such as sets and numbers. Rather, it seems that Gödelian intuition is really about concepts.

In this chapter, in trying to answer the problems raised above, we will examine Gödelian intuition. In the first section, we will make clear what Gödelian intuition really is. This clarification will also reacknowledge the importance of concepts in Gödelian Platonism. In the second section, we will shortly examine one of the advantages of accepting Gödelian intuition as an epistemological standpoint in mathematics. This examination will also reveal a problem in Gödelian Platonism. In the third section, we will examine the supplement to "What is Cantor's continuum problem" with special attention to the relation between the thought of Gödel, Kant, and Husserl. This examination will deepen our understanding about Gödelian intuition and Gödelian Platonism.

1 Mathematical intuition according to Gödel

In the previous chapter, we have made clear the importance of concepts in Gödelian Platonism. In short, Gödel thinks mathematics is inquiry concerning mathematical concepts. However, even if we admit this Gödelian conception about mathematical activity, there still remains a serious problem: what connects us to concepts? If there is no epistemological connection between us and concepts, the considerable part of Gödelian Platonism is just pie

[5][Chihara 1990, pp. 15-21].

in the sky.

Traditionally, in the foundational studies of mathematics, the ontological and epistemological aspects of mathematics do not seem to have had been adequately dealt with. More precisely, these aspects had been carefully removed from the discourse of the foundations of mathematics. For example, formalists thought that mathematics is a mere game of symbols.[6] In this conception, it is meaningless to talk about the ontological (and consequently epistemological) status of symbols because a symbol can mean anything and therefore it does not have any particular meaning by itself. On the other hand, constructivists see mathematics as nothing but creations of our minds. Thus, for constructivists, to look inside our minds is enough to communicate with mathematical objects. However, for realists like Gödel who believe in the objective existence of mathematical objects and think that mathematics is a meaningful activity, these views concerning the ontological and epistemological aspects of mathematics are not admissible at all. As to the ontological status of mathematics, Gödel tried to defend his realist view mainly by appealing the equi-supportive and consequentialist arguments.[7] As to the epistemological problem, it is said that Gödel appeals to what is called *mathematical intuition*.

Strange as it might seem, despite having long been regarded as one of the most notable, and perhaps most notorious, aspects of Gödelian Platonism even before the publication of Gödel's *Collected Works*, mathematical intuition had not been a fundamental element in

[6]This is, needless to say, too much simplified a view about formalism. As we briefly mentioned in the footnote 5 of the previous chapter, Hilbert, the founder of formalism, seems to have had a rather realist conception about mathematics in that he believed that every mathematical problem is solvable in some way.

[7]As to the equi-supportive and consequentialist argument of Gödel, see the section 2 of Chapter 1.

Gödelian Platonism at least until 1953 when Gödel started writing "Is mathematics syntax of language?" According to [Parsons 1995], the term "intuition" appears only three times in [Gödel 1944]. The first appearance is used within quotation marks and supposedly thought of as the quotation from Hilbert.[8] The second, about which Parsons says that "one of the most often quoted remarks in the paper," is used as meaning "common-sense assumptions of logic."[9] The third is the same as the second.[10] In [Gödel 1947], there is only one appearance of the term in the paragraph which argues against constructivism. The term is used in expressing a constructivist view and therefore does not have anything with Gödel's own view.[11] In [Gödel 1951], the term does not appear at all.[12]

It is in [Gödel 1953/59] that Gödel finally starts talking about mathematical intuition as playing an important role in his Platonism. The very first appearance of the term "intuition" in the proper sense is in the following footnote to the words "intuitive content."

> The existence, as a psychological fact, of an intuition covering the axioms of classical mathematics can hardly be doubted, not even by adherents of the Brouwerian school, except that the latter will explain this psychological fact by the circumstance that we are all subject to the same kind of errors if we

---

[8] [Gödel 1944, p. 121].

[9] [Gödel 1944, p. 124].

[10] [Gödel 1944, p. 138].

[11] [Gödel 1947, p. 180].

[12] As Parsons points out, Gödel mentions the perception of mathematical objects in [Gödel 1951]. However, as in the case of [Gödel 1944], Gödel simply presupposes such a perception and does not develop any theory about it.

> are not sufficiently careful in our thinking.[13]

As seen above, Gödel thinks that it is undeniable even for stubborn constructivists that there is "an intuition covering the axioms of classical mathematics," that is, an intuition that all axioms of classical mathematics should be true. However, Gödel does not think that the fact that we have such an intuition about mathematics (or mathematical axioms) is enough to assure the security of mathematics. If having such an intuition is enough for paradox-free mathematics, doing mathematics would be rather an easy task.[14] However, in fact, it is always possible that we mistakenly think of incorrect axioms (or propositions) as true. Consequently, Gödel carefully avoids saying that having mathematical intuition means having mathematical knowledge. We need something other than intuition to get to mathematical truth. In this connection, Gödel says the following.

> [I]f mathematical intuition and the assumption of mathematical objects or facts is to be dispensed with by means of syntax, it certainly will have to be required that the use of the "abstract" and "transfinite" concepts of mathematics, which cannot be understood or used without mathematical intuition or assumption of their properties, be based on considerations about finite combinations of symbols.[15]

---

[13] [Gödel 1953/59-III, p. 338].

[14] It seems that those who think of Gödelian intuition as mystical or mysterious have in mind this kind of unerring intuition which unfailingly provides us with (mathematical) truth. However, as discussed in the body, Gödelian intuition is not infallible.

[15] [Gödel 1953/59-III, p. 341].

At least two points should be noted about the view expressed here. First, both mathematical intuition and the existential assumption of mathematical objects are necessary to underwrite mathematical truth. Second, they are also necessary to understand and use the abstract and transfinite concepts. As clearly seen from these points, mathematical intuition cannot be an infallible faculty which directly apprehends mathematical objects. If mathematical intuition were such a faculty, there would be no need for the assumption of mathematical objects in addition to mathematical intuition.

In the above, it is said that mathematical intuition is not what gives us direct knowledge of mathematical reality. Then, what is mathematical intuition, really? Note that mathematical intuition is compared to syntactical conception of mathematics in the above quotation. Actually, right before the above quotation, Gödel says that

> the original purpose and the chief interest of the syntactical interpretation refer to the question as to whether (in particular in the application of mathematics) it can *replace the belief in the correctness of mathematical intuition*.[16]

According to Gödel, the syntactical interpretation of mathematics is the view which asserts that

> mathematics can completely reduced to (and in fact *is* nothing but) syntax of language. I.e., the validity of mathematical theorem consists solely in their being consequences of certain syntactical conventions about the use of symbols. . . .[17]

---

[16][Gödel 1953/59-III, pp. 340-341]; italics in the original.

[17][Gödel 1953/59-III, p. 335]; italics in the original.

In short, both the syntactical interpretation of mathematics and mathematical intuition are about the validity of mathematical propositions. From the syntactical point of view, we regard a proposition as being true only when it can be deducible from a certain set of tautologies by means of syntactical transformation rules. On the other hand, mathematical intuition gives us the conviction about the validity of mathematical propositions that

> if these sentences [i.e., mathematical propositions] express observable facts and were obtained by applying mathematics to verified physical laws (or if they express ascertainable mathematical facts), then these facts will be brought out by observation (or computation).[18]

The question here is: Can the syntactical conception of mathematics give us the same conviction as to mathematical propositions as mathematical intuition supposedly gives? First, Gödel admits that we can arrive at the same mathematical proposition, if it is true, in either way. However, Gödel asserts that we cannot give any credence to a proposition if it is attained by means of syntactical transformation rules because of his incompleteness theorem. Why does the incompleteness theorem, however, prevent us from giving credence to a proposition derived by syntactical rules? Let us think in the following way.

In order to give credence to a proposition derived from a syntactical system, we have to make sure that such a system is consistent because any proposition can be derived from an inconsistent system. However, according to the incompleteness theorem, if a system is powerful enough to develop the theory of arithmetic and if it is actually consistent, the consistency of the system cannot be shown within the system itself. Needless to say, any syn-

[18] [Gödel 1953/59-III, p. 340].

tactical system must be consistent and powerful enough to develop the theory of arithmetic because it is supposed to cover the whole of mathematics. Therefore, there is no assurance that a syntactical system is consistent and then we cannot be perfectly sure whether a proposition is derivable from such a syntactical system.[19] This is why, Gödel maintains, we have to appeal to mathematical intuition to have convictions about the validity of mathematical propositions.

In the fifth manuscript of "Is mathematics syntax of language?," Gödel appeals to the analogy of physical perception again as in [Gödel 1944]. But this time, unlike [Gödel 1944], Gödel explicitly compares mathematical intuition to physical perception.

> The similarity between mathematical intuition and a physical sense is very striking. It is arbitrary to consider "This is red" an immediate datum, but not so to consider the proposition expressing modus ponens or complete induction. . . . For the difference, as far as it is relevant here, consists solely in the fact that in the first case a relationship between a concept and a particular object is perceived, while in the latter it is a relationship between concepts.[20]

Gödel says in the above quotation that mathematical intuition and a physical sense are similar in that both have the "perception" of concepts.[21] For example, the proposition "This

[19]Accordingly, if we are to work within some formal system, we have to first *believe* the consistency of the system. But how? Gödel must respond that "by mathematical intuition."

[20][Gödel 1953/59-V, p. 359].

[21]This similarity between mathematical intuition and a physical sense plays an important role in the issue about the applicability of mathematics to physical sciences. We will return to this point in the later section.

is red" contains the concept "being red." However, the truth of the proposition depends on the indexical "This." In other words, the truth of the proposition is contingent. Thus, the proposition cannot be regarded as "an immediate datum," that is, necessary truth. On the other hand, modus ponens can be regarded as expressing necessary truth because its truth does not depend on its constituents. But why are propositions expressing logical laws such as modus ponens and complete induction thought of as containing "a relationship between concepts"?

As Parsons points out,[22] it is not altogether clear why Gödel thinks that what is "perceived" in "the proposition expressing modus ponens or complete induction" is "a relationship between concepts." Parsons seems to think that Gödel thinks this because modus ponens contains a relationship between its constituents p, $p \rightarrow q$, and q. However, in this interpretation, it seems that even the proposition "This is red" can be interpreted as expressing a relationship between two concepts "This" and " . . . is red." It seems more relevant, at least in this case, to interpret this relationship as that between $\{p_1, p_1 \rightarrow q_1 \vdash q_1\}$, $\ldots$, $\{p_n, p_n \rightarrow q_n \vdash q_n\}$. In short, the objects of mathematical intuition are the universal validity expressed in propositions, not each individual concept.

The above interpretation of mathematical intuition, that it is "something like a perception" the objects of which are the universal validity, or the "immediate givenness," of a relationship between concepts seems, however, to raise other problems for Gödelian Platonism. First, as already seen, Gödel does not think of mathematical intuition as providing us with direct knowledge of mathematics, but as its source. However, in the above interpretation of mathematical intuition, it can be thought of almost as providing direct knowledge

[22][Parsons 1995, p. 62].

of mathematics. As to this point, we should note that Gödel uses the word "conviction" to characterize what mathematical intuition gives us. In order for such a conviction to become "knowledge," we still have to have something other than mathematical intuition.[23] Second, there seems to be another kind of "intuition" which is specialized for "perceiving" concepts, not their relations. Then, it follows that there are different epistemological functions in each case. However, if we posit (at least) two kinds of "intuition," then there are two disjoint realms about which intuition is used. This does not seem to conform with Gödel's holistic conception of physical/mathematical theories discussed in the section 5 of Chapter 1. Because this point is closely related to what we will examine in the next section, we will return to it then.

Taking the above considerations into account, let us summarize our arguments about mathematical intuition so far. First of all, mathematical intuition is primarily about relations between concepts. However, we should recall here that a concept is essentially relational, especially in mathematics.[24] In other words, in a seemingly individual concept, there are other concepts behind it. Let us take up the concept "set" as an example as before. The concept "set" is actually the collection of axioms each of which in turn represents a concept (or concepts). With this conception of "concept," we can give more complete account of why the object of mathematical intuition can be interpreted as the validity or consistency of the collection of concepts. As we said, the concept "set" is actually the collection of axioms.

---

[23]In this reading, mathematical intuition can be thought of as a kind of Ariadne's thread. However, according to Parsons, Gödel attributes this conception of mathematical intuition to Carnap and regards it as irrelevant ([Parsons 1995, p. 61]). Unfortunately, we could not figure out where Gödel mentions that.

[24]See the section 3 of the previous chapter.

In order for the concept "set" to be meaningful, it must be a consistent concept. In other words, no inconsistent concept can be derived from axioms or concepts which comprise the concept "set." Thus, when we justly apprehend a concept by mathematical intuition, we also apprehend the consistency of the collection of concepts which comprise that concept.[25]

It would be helpful for understanding mathematical intuition better to recall the analogy between mathematics and physical sciences in [Gödel 1944]. Gödel argues the analogy as follows.

> They [objects of logic and mathematics] are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about "data," i.e., in the latter case the actually occurring sense perceptions.[26]

Needless to say, "data" in the above quotation are relations of concepts in the case of mathematics. As repeatedly said, however, these data or mathematical intuition which supposedly access these data can be wrong. To make a satisfactory theory of mathematics, we need something other than mathematical intuition. That is, having mathematical intuition alone is not enough to have mathematical knowledge. First of all, we need the assumption

[25]Note that the possibility that mathematical intuition makes mistake is not excluded as we pointed out earlier. However, this possibility does not mean at all that we can never have a satisfactory mathematical theory just as the possibility of erring in perceiving physical objects does not mean the impossibility of physical sciences.

[26][Gödel 1944, p. 128].

of the existence of mathematical objects in addition to mathematical intuition to get a satisfactory system of mathematics. Sometimes we might even appeal to inductive methods to ensure the correctness and appropriateness of mathematical data because mathematical intuition is sometimes wrong. Nevertheless, the necessity of these additional requirements does not imply the dispensability of mathematical intuition. For example, in the case of physical sciences, we need to presuppose the objective existence of physical objects and, more importantly, the universal validity of physical laws. However, to construct any physical theory, we still have to first of all perceive something physical, even if there is the possibility that the perception errs. (If we do not perceive anything physical at all, why should we bother making a physical theory?) Similarly, in the case of mathematics, we need mathematical intuition in order to get "data" as starting points to construct a theory of mathematics.

Actually, the above interpretation of the analogy between physical sciences and mathematics is not perfectly adequate as that of the analogy in [Gödel 1944].[27] However, as we have hopefully shown, the 1944 viewpoint of Gödel as to the analogy is consistent with the conception of mathematical intuition discussed here. Moreover, the analogy between mathematics and physical sciences in the light of mathematical intuition brings us to another aspect of relations between mathematics and physical sciences: the applicability of mathematics to physical sciences. In the next section, we will examine this aspect of Gödel's thought.

---

[27]The main point of this analogy in [Gödel 1944] is the existence of mathematical objects, not mathematical intuition. As to the interpretation faithful to the original context, see the section 2 of Chapter 1.

2 The applicability of mathematics

As we have seen, Gödel argues that it is only when propositions (or laws) of physical sciences have mathematical elements as their constituents that these propositions can be said to have contents.[28] On the other hand, Gödel believes that there are two separate worlds, that is, "the world of things and of concepts."[29] Based on this conception of worlds, Gödel says:

> [W]hile through sense perception we know particular objects and their properties and relations, with mathematical reason we perceive the most general (namely the "formal") concepts and their relations, which are separated from space-time reality insofar as the latter is completely determined by the totality of the particularities without any reference to the formal concepts.[30]

On the surface, the above quotation seems contradict with what Gödel says about the interrelatedness between mathematics and physical sciences. Moreover, in general, it seems clearly true that mathematics plays an extremely important role in physical sciences. How is it possible at all to apply mathematics to physical theories if mathematics and "space-time reality" are two disjoint worlds?

For those who interpret mathematics as a matter of transformation of contentless symbols, that is, for conventionalists and formalists, the above question is rather easy to answer. In the conventionalist or formalist interpretation, mathematics is a mere useful

---

[28] As to this point, see the section 5 of Chapter 1.

[29] [Gödel 1951, p. 321].

[30] [Gödel 1953/59-III, p. 354]. We should have made clear what "mathematical reason" means here, but we simply identify it with mathematical intuition for now.

device which simplifies inference steps between empirical propositions. As to this point, Field points out that in applying mathematics to physical theories it is necessary for the resulting theory, i.e., the theory to which mathematics was applied, to be a conservative extension of the original theory, because otherwise it follows that mathematics adds some contents to the original theory and then that mathematics has contents.[31]

Gödel, needless to say, does not take the conventionalist/formalist route in explaining the applicability of mathematics to physical theories. Gödel says:

> What mathematics adds to the physical laws, it is true, are not any new properties of physical reality, but rather properties of the *concepts* referring to physical reality — to be more exact, of the concepts referring to combinations of things.[32]

Gödel partially agrees with conventionalists and formalists as to the relation between mathematics and physical sciences; how physical reality exists is independent of, or separated from, how mathematical reality exists. Thus, it is impossible, Gödel admits, that mathematics adds something which can be alter the mode of existence to physical objects. Nevertheless, as seen above, Gödel asserts that mathematics adds "properties of the *concepts* referring to combinations of things" to the physical laws. However, what are these properties?

First, note that what is said in the above quotation is very similar to what is said in [Gödel 1944] as to the relation between mathematics and physical sciences.[33] In [Gödel 1944],

---

[31] [Field 1980, pp. 8-11].

[32] [Gödel 1953/59-III, p. 349]; italics in the original.

[33] [Gödel 1944, p. 128]. For the quotation of the relevant part, see page 11 of this thesis.

it is said that just perceiving a physical object or an event containing physical objects is not enough to form a theory about that object or event. In order to form a satisfactory theory about the physical world, we need something beyond that world. For example, suppose that we perceive an object's falling. A mere perceiving of an object's falling of course does not establish any law about the object which fell or the event of falling. After several observations of the falling-event, we come to have the *conviction* that in ordinary circumstances on earth any physical object falls when it is released from above. In other words, we believe that propositions $F(p_1)$, ..., $F(p_n)$ hold.[34]

Recall here what we said in the previous section about the similarity between mathematical intuition and physical sense. In talking about the similarity, Gödel said that the object of mathematical intuition is "a relationship between concepts."[35] And we argued that this "relationship between concepts" should be interpreted as expressing some kind of universal validity. This interpretation is in fact underwritten by what Gödel says about "properties of the concepts referring to combinations of things."

> [I]t is perfectly possible that properties of concepts (if they contain universal quantifiers) may *not* follow from the definitions or the meanings of the terms . . . but still may be knowable in the same sense as laws of nature.[36]

[34]Here, p stands for an element of the set of physycal objects (including possible ones) and $F(x)$ stands for "in ordinary circumstances on earth x falls when it is released from above."

[35][Gödel 1953/59-V, p. 359]. For the quotation, see p. 69 of this thesis.

[36][Gödel 1953/59-III, p. 349].

Here, properties of concepts *with* universal quantifiers are said not to "follow from the definitions or the meanings of the terms." Taking into account the context in which the above quotation appears, "the definitions or the meanings of the terms" can be thought of as descriptions or sense perception of physical reality. Thus, in order for a mere physical fact to become a part of physical laws, it is necessary to add a universal aspect to the fact by mathematical intuition.

The above argument about the applicability of mathematics remind us of the Kantian argument concerning knowledge of the external world. Roughly speaking, the Kantian argument is as follows. First, according to Kant, we cannot have direct knowledge of external objects. The images of objects perceived by us are always and already filtered by our internal faculties. One of such faculties is called *understanding*, which plays an important role in theory formation. By understanding, we can posit laws (causalities) which hold between external objects (or events) which are seemingly in random disposition without the help of understanding.

The similarity between Gödel and Kant is clear. For Gödel as well, mere data (in the above term, the images of objects or events) are not enough for forming theories. In addition to such data, we need another element which establishes causal relations between data. This additional element is, for Gödel, mathematical intuition. As already seen, by mathematical intuition we can grasp and underwrite the universal aspects in mathematics and physical theories.

On the other hand, in addition to the similarity stated above, there are of course differences between Gödel and Kant. However, to see the difference properly, we have to examine a relationship between Gödel and another German prominent thinker Edmund Husserl. In the next section, by reading closely the supplement to [Gödel 1947] about which we have intentionally kept silent, we will make clear this relationship.

3 Gödel, Kant, and Husserl

When his [Gödel 1947] was reprinted in [Benacerraf and Putnam 1964], Gödel made over one hundred alterations in the original [Gödel 1947], most of which are stylistic, and added two sections at the end of the paper. In this section, we will concentrate on examining one of these additional sections titled "Supplement to the second edition."[37] In doing so, we will shed light on the intellectual relationship of Gödel to Kant and Husserl as well as on Gödel's latest thought.

After discussing the difference between the independence of the parallel postulate in geometry and of the continuum hypothesis in set theory, Gödel devotes the rest of the supplement almost entirely to the defense of his Platonism. Gödel starts his defense as follows.

> [D]espite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.

This passage, along with the analogy between mathematics and physical sciences in [Gödel 1944], has been the target of criticism. For example, Chihara says in [Chihara 1990]

[37]Another additional section is just a short remark on the result of Paul Cohen which showed the independence of the continuum hypothesis from axioms of set theory.

that "I do not find Gödel's reasoning on this matter very convincing"[38] because some axioms which are usually supposed in ZF set theory — Chihara brings up the axiom of regularity as an example[39] — might not hold in other systems of set theory such as Quine's NF set theory or non-well-founded set theory. To respond to Chihara's criticism, we should recall here what Gödel said about the objects of mathematics and mathematical intuition.

First, for Gödel, the objects of mathematics are not limited to classes such as "sets" or "numbers." Rather, as we have seen, concepts are the more important objects of mathematical intuitions than classes in the Gödelian conception. Consequently, the phrase "the objects of set theory" first of all refers to the concepts of set theory. On the other hand, concepts should be understood in relation to other concepts. Especially, for mathematical intuition, it is the relations among concepts that count. Therefore, based on these interpretations of the terms, Chihara's interpretation does not do justice to Gödel's thought because in his interpretation Chihara isolates an axiom from other axioms and raises the false (at least from the Gödelian point of view) problem whether or not the axiom is true *per se*. However, it is not an isolated individual axiom the truth of which we inquire about, but rather a collection of axioms.

In the next paragraph, which Parsons describes as "possibly the most difficult and obscure passage in Gödel's finished philosophical writing," Gödel explicitly compares his thought with Kant's. He begins the paragraph with a note about mathematical intuition.

It should be noticed that mathematical intuition need not be conceived of as

[38][Chihara 1990, p. 17]

[39][Chihara 1990, pp. 18-19].

> a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of these objects on the basis of something else which *is* immediately given.[40]

We have already encountered a similar thought. In the first section of this chapter, we showed, by analyzing what Gödel says in [Gödel 1953/59], that to apprehend mathematical truth we need something other than mathematical intuition: the assumption of mathematical objects.[41] However, this time Gödel brings up the analogy between mathematical intuition and sense perception in this regard.

> [T]his something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself. . . . Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas.[42]

As in the case of mathematical intuition, even sense perception needs something other than mere sensations to form the idea of physical objects. We need first of all the idea of the object itself as what is immediately given. It is clear that Gödel relies here on Kantian

---

[40][Gödel 1964, p. 268]; italics is in the original.

[41]See pp. 66-67 of this thesis.

[42][Gödel 1964, p. 268]; italics is in the original.

thought which we saw in relation to the applicability of mathematics.[43] Moreover, in the last sentence in the above quotation, Gödel asserts that mathematics and physical experience share something abstract. As we examined in the last section, because of these abstract elements in empirical ideas which are supposed to be apprehend by mathematical intuition, we can apply mathematics to empirical sciences. And Gödel, unlike Kant, does not think of these abstract elements in empirical ideas as something subjective. Gödel says:

> It is by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality.[44]

This should be considered in light of what Gödel says in [Gödel 1961]. There, in criticizing Kant for "the lack of clarity and the literal incorrectness" in his writing, Gödel refers to phenomenology by which we can avoid "both the death defying leaps of idealism into a new metaphysics as well as the positivistic rejection of all metaphysics."[45] In short, Gödel thinks that with the help of phenomenology we can confirm the objective existence of abstract elements in empirical ideas and the accessibility to such abstract elements.[46]

---

[43]See p. 77 of this thesis.

[44][Gödel 1964, p. 268].

[45][Gödel 1961, p. 387].

[46]In fact, Gödel already criticized this tendency in Kantian thought to deny the objectivity of abstract

However, oddly enough, Gödel does not mention phenomenology at all in [Gödel 1964].[47]

Actually, throughout his writings, whether published or not, Gödel offered few details about how exactly phenomenology could be useful for his purpose. The most explicit mention to phenomenology is in [Gödel 1961]. As we have already seen in the last section of chapter 1, there Gödel regarded phenomenology as enabling us to make clear the meanings in mathematical terms "by directing our attention in a certain way . . . onto our own acts in the use of these concepts, onto our power in carrying out our acts."[48] Following this, Gödel also make important remarks about phenomenology.

> [O]one must keep clear in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is . . . a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other

---

elements in empirical ideas in [Gödel 1946/49]. He says: "Unfortunately, whenever this fruitful viewpoint of a distinction between subjective and objective elements in our knowledge (which is so impressively suggested by Kant's comparison with the Copernican system) appears in the history of science, there is at once a tendency to exaggerate it into a boundless subjectivismm whereby its effect is anulled. Kant's thesis of the unknowability of the things in themselves is one example." ([Gödel 1946/49, pp. 257-258]).

[47]Actually, Gödel mentioned the relation of his thought to phenomenology in a draft of the supplement. "Perhaps a further development of phenomenology will, some day, make it possible to decide questions regarding the soundness of primitive terms and their axioms in a completely convincing manner" (cited from [van Atten and Kennedy 2003, p. 466]). Van Atten and Kennedy guess that Gödel left this sentence out because of his fear of positivist attacks.

[48][Gödel 1961, p. 383].

basic concepts hitherto unknown to us.[49]

Roughly, Gödel considers phenomenology as a way to reflect the flow of our consciousness when we use concepts. In doing so, Gödel believes, we can attain "a new state of consciousness" in which we can make the basic concepts clear and even find new ones. And this "reflection to consciousness" clearly refers to the intentionality of consciousness in Husserl's phenomenology.

Intentionality is, needless to say, one of the most important elements in Husserl's thought. It is roughly interpreted as an act of consciousness which always directs consciousness toward something. In this act of consciousness, a mind which is supposedly identified with consciousness and an object toward which consciousness is directed are indissolubly tied. Thus, a conundrum which arose from Cartesian dualism may be thought to disappear. In a sense, Husserlian intentionality integrates something psychic and physic.

As a corollary to the characteristics of intentionality explained above, it follows that phenomenology does not bother with the problem of the reality of objects toward which consciousness directs itself. Rather, for phenomenology, what is important is how objects are presented to consciousness, that is, the meaning of objects. Consequently, it is a focal point for phenomenology to make the meaning of objects clear.

In addition to intentionality, another element from Husserl which seemingly has importance for Gödel is intuition. Husserl distinguishes two kinds of intuition: sensuous and categorial. Through sensuous intuition, we can get sense data of something. On the other hand, categorical intuition enables us to bind disorganized sense data into one united entity.

[49][Gödel 1961, p. 383].

In other words, categorical intuition, like Kantian understanding, gives one definite concept to a manifold of sense data.

The similarity between Husserl and Gödel should be clear from the above remarks about intentionality and intuition. What Gödel asserts as to how we can clarify the meanings of mathematical terms is almost identical to the above explanation of Husserlian intentionality. The similarity in their conception of intuition is also striking. For both Gödel and Husserl, intuition is an integrating power.

As we have seen above, in at least two important aspects — how to clarify the meaning and mathematical intuition — Gödel seems to be heavily influenced by Husserl. Although the above comparison is very sketchy, it hopefully sheds new light on how Gödelian Platonism should be interpreted. That is, Gödelian Platonism might be better understood from the Kantian-Husserlian point of view rather than from the analytical point of view.

Concluding Remarks

In this thesis, we tried to show the plausibility of Gödelian Platonism, which has long been regarded as naïve and amateurish by many scholars. To accomplish this purpose, we concentrated on two important aspects of Gödelian Platonism: concepts and mathematical intuition.

In Chapter 2, we showed that concepts are first of all relational entities. They should not be considered in isolation. Thus, for example, the concept "set" should be thought as the aggregation of concepts which can be expressed through axioms such as that of extensionality or that of infinity. For Gödel, clarifying concepts is the key to advancing mathematical knowledge and solving mathematical problems. However, in relation to concepts, a serious problem arises: how can we epistemically access such concepts? We dedicated Chapter 3 to addressing this problem.

We showed in Chapter 3 that we access concepts by what Gödel calls *mathematical intuition* and argued that mathematical intuition is not a mystical and mysterious faculty which can give us direct knowledge of mathematics. It is not the ability to grasp each concept in isolation. Rather, it is the ability to ascertain the consistency of concepts. Moreover, by presupposing mathematical intuition and conceptual aspects in physical entities, we can explain why mathematics is applied to physical sciences.

Besides these fundamental elements, we also extracted other characteristics of Gödelian Platonism from what Gödel says, mostly in Chapter 1. First, we showed Gödel's characteristic method defending his Platonism. Gödel had never defended it by dogmatically asserting the existence of mathematical objects. Whenever he defended the existence of mathematical objects, he did so by arguing that if we admit the existence of physical objects, we have to admit the existence of mathematical objects as well. In a sense, for Gödel, physical and mathematical worlds share something abstract. Second, as a result of Gödel's conception of the mathematical world as objective, he asserted that we can use inductive or experimental methods in mathematics.

Although we believe that we succeeded in achieving our primary objective, that is, showing the plausibility of Gödelian Platonism, there are of course many aspects of Gödelian Platonsim which we could not deal with in this thesis.

First of all, we could not argue about the objective existence of mathematical objects *per se*. In other words, we mainly concentrated on the epistemological aspect of Gödelian Platonism and on understanding properly Gödel's metaphysical claims. Although our omission of this kind of argument from our agenda might be justified by the fact that Gödel himself did not argue the existence of mathematical objects *per se*, it should be noted that the problem about the existence of mathematical objects is an important topic in the philosophy of mathematics.

Second, we could not argue about another important aspect of Gödelian Platonism: the contribution of Gödel's Platonist conception of mathematics to his actual mathematical achievements. In a letter to Hao Wang, Gödel said that he succeeded in proving the completeness theorem because of his Platonist temperament while Skolem failed because of his finitist tendency.[1] Although it would be very exciting to examine what Gödel said in this regard, to amply do so definitely needs another paper.

Lastly, we could only briefly mention the relation between Gödel and Husserl. How-

[1][Gödel 2003, p. 397].

ever, as with the above issue, to fully examine their relation needs another — perhaps very long — paper.

As we hopefully showed in this thesis, Gödelian Platonism is not as naïve and amateurish as many scholars still think. Rather, it was a sophisticated view developed over long time. Moreover, from the late 1950s, Gödel devoted more time to the study of philosophy than mathematics. His Platonism is not the fruit of a mathematician's diversion at all. It deserves serious consideration. Moreover, as we mentioned above, there are a plenty of unexplored themes in Gödelian Platonism. We can still learn much from it.

Bibliography

[van Atten and Kennedy 2003] van Atten, M. and Kennedy, J., "On the philosophical development of Kurt Gödel," *The Bulletin of Symbolic Logic*, **9**, no. 4 (2003), 425–476.

[Benacerraf and Putnam 1983] Benacerraf, P. and Putnam, H. (eds.), *Philosophy of Mathematics: Selected Readings*, Second Edition, Blackwell (1983).

[Bernays 1946] Bernays, P., Review of [Gödel 1944], *The Journal of Symbolic Logic*, **11**, 75–79.

[Carnap 1937] Carnap, R., *The Logical Syntax of Language*, Routledge & Kegan Paul (1937).

[Cassou-Noguès 2005] Cassou-Noguès, P., "Gödel and 'the objective existence' of mathematical objects," *History and Philosophy of Logic*, **26** (2005), 211–228.

[Chihara 1973] Chihara, C., *Ontology and the Vicious Circle Principle*, Cornell University Press (1973).

[Chihara 1982] Chihara, C., "Gödelian thesis regarding mathematical objects: do they exist? and how can we perceive them?," *Philosophical Review*, **11** (1982), 211–228.

[Chihara 1990] Chihara, C., *Constructibility and Mathematical Existence*, Oxford University Press (1990).

[Colyvan 2001] Colyvan, M., *The Indispensability of Mathematics*, Oxford University Press (2001).

[Davis 1965] Davis, M. (ed.), *The Undecidable: Basic Papers on Undecidable Propositions, Unsolvable Problems and Computable Functions*, Raven Press (1965).

[Davis 2005] Davis, M., "What did Gödel believe and when did he believe it?", *The Bulletin of Symbolic Logic*, **11**, no. 2 (2005), 194–206.

[Dawson 1997] Dawson, J. W., *Logical Dilemmas: The Life and Work of Kurt Gödel*, A K Peters (1997).

[Duhem 1914] Duhem, P., *La théorie physique: son objet, et sa structure*, Second Edition, M. Rivière (1914).

[Field 1980] Field, H., *Science Without Numbers: A Defense of Nominalism*, Princeton University Press (1980).

[Føllesdal 1999] Føllesdal, D., "Gödel and Husserl" (1999) in [Petitot et al. 1999], pp. 385–400.

[Gödel 1933] Gödel, K., "The present situation in the foundations of mathematics" (1933) in [Gödel 1995], pp. 45–53.

[Gödel 1947] Gödel, K., "What is Cantor's continuum problem?" (1947) in [Gödel 1990], pp. 119–141.

[Gödel 1951] Gödel, K., "Some basic theorems on the foundations of mathematics and their implications" (1951) in [Gödel 1995], pp. 304–323.

[Gödel 1953/59-III] Gödel, K., "Is mathematics syntax of language?" (1953), version III in [Gödel 1995], pp. 334–355.

[Gödel 1953/59-V] Gödel, K., "Is mathematics syntax of language?" (1953), version V in [Gödel 1995], pp. 356–362.

[Gödel 1961] Gödel, K., "The modern developments of the foundations of mathematics in the light of philosophy" (1961) in [Gödel 1995], pp. 375–384.

[Gödel 1964] Gödel, K., "What is Cantor's continuum problem?" (1964), revised and expanded version of [Gödel 1947] in [Gödel 1990], pp. 119–141.

[Gödel 1990] Gödel, K., *Collected Works Volume II*, S. Feferman *et al.* (eds.), Oxford University Press (1990).

[Gödel 1995] Gödel, K., *Collected Works Volume III*, S. Feferman *et al.* (eds.), Oxford University Press (1995).

[Gödel 2003] Gödel, K., *Collected Works Volume IV*, S. Feferman *et al.* (eds.), Oxford University Press (2003).

[Hilbert 1928] Hilbert, D., "Die Grundlagen der Mathematik," *Abhandlungen aus dem Seminar der Hamburgischen Universität*, **6**, 65-85. English translation in [van Heijenoort 1967], pp. 464-479.

[Husserl 1913/21] Husserl, E., *Logische Untersuchungen*, 2 volumes, Second Edition, Niemeyer (1913/21). Translated as [Husserl 1970].

[Husserl 1913] Husserl, E., *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Book 1, Nijhoff (1913). Translated as [Husserl 1982].

[Husserl 1950] Husserl, E., *Die Idee der Phänomenologie*, Nijhoff (1950). Translated as [Husserl 1964].

[Husserl 1964] Husserl, E., *The Idea of Phenomenology*, Nijhoff (1964).

[Husserl 1970] Husserl, E., *Logical Investigations*, 2 volumes, Humanities Press (1970).

[Husserl 1984] Husserl, E., *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy*, Book 1, Nijhoff (1984).

[Kleene 1952] Kleene, S., *Introduction to Metamathematics*, Woltersnoordhoff Publishing and North-Holland Publishing Company (1952).

[Kockelmans 1994] Kockelmans, J. J., *Edmund Husserl's Phenomenology*, Purdue University Press (1994).

[Maddy 1990] Maddy, P., *Realism in Mathematics*, Oxford University Press (1990).

[Maddy 1997] Maddy, P., *Naturalism in Mathematics*, Oxford University Press (1997).

[Martin 2005] Martin, D., "Gödel's conceptual realism," *The Bulletin of Symbolic Logic*, **11**, no. 2 (2005), 207–224.

[Parsons 1995] Parsons, C., "Platonism and mathematical intuition in Kurt Gödel's thought," *The Bulletin of Symbolic Logic*, **1**, no. 1, 44–74.

[Petito et al. 1999] Petitot, J. et al. (eds.), *Naturalizing Phenomenology: Issues in Contemporary Phenomenology and Cognitive Science*, Stanford University Press (1999).

[Poincaré 1906] Poincaré, H., "" in [Poincaré 1947] with modification, pp. 152–214.

[Poincaré 1947] Poincaré, H., *Science et méthode*, Flammarion (1947).

[Potter 2001] Potter, M., "Was Gödel a Gödelian Platonist?", *Philosophia Mathematica*, **9** (2001), 331–346.

[Quine 1948] Quine, W. V. O., "On what there is" (1948) in [Quine 1953], pp. 1–19.

[Quine 1951] Quine, W. V. O., "Two dogmas of empiricism" (1951) in [Quine 1953], pp. 20–46.

[Quine 1953] Quine, W. V. O., *From a Logical Point of View*, Harvard University Press (1953).

[Ramsey 1926] Ramsey, F., "The Foundations of mathematics" (1926) in [Ramsey 1931], pp. 1–61.

[Ramsey 1931] Ramsey, F., *The Foundations of Mathematics and Other Logical Essays*, Routledge & Kegan Paul (1931).

[Russell 1903] Russell, B., *The Principles of Mathematics*, Allen and Unwin (1903).

[Russell 1920] Russell, B., *Introduction to Mathematical Philosophy*, Second Edition, Allen and Unwin (1920).

[Russell 1957] Russell, B., *Mysticism and Logic*, Doubleday (1957).

[Russell and Whitehead 1925] Russell, B. and Whitehead, A. N. *Principia Mathematica*, vol. 1, Second Edition, Cambridge University Press (1925).

[Stewart and Mickunas 1990] Stewart, D. and Mickunas, A., *Exploring Phenomenology: A Guide to the Field and Its Literature*, Second Edition, Ohio University Press (1990).

[Tieszen 2002] Tieszen, R., "Gödel and the intuition of concepts," *Synthese*, **133** (2002), 363–391.

[Turing 1937] Turing, A., "On computable numbers, with an application to the Entscheidungsproblem" (1937) in [Davis 1965], pp. 116–154.

[van Heijenoort 1967] van Heijenoort, J. (ed.), *From Frege to Gödel*, Harvard University Press (1967).

[Wang 1962] Wang, H., *A Survey of Mathematical Logic*, Science Press (1962). Reprinted as *Logic, Computers and Sets*, Chelsea (1970).

[Wang 1974] Wang, H., *From Mathematics to Philosophy*, Routledge & Kegan Paul (1974).

[Wang 1987] Wang, H., *Reflections on Kurt Gödel*, The MIT Press (1987).

[Wang 1996] Wang, H., *Logical Journey*, The MIT Press (1996).

[Wittgenstein 1922] Wittgenstein, L., *Tractatus Logico-Philosophicus*, Routledge & Kegan Paul (1922).

[Woodin 2001] Woodin, W. H., "The Continuum Hypothesis, I & II," *Notices, American Mathematical Society*, **48**-6 (2001), 567–576 & **8**-7 (2001), 681–690.